Extra credit: uncertainty of conjugate variables:

Consider a "normally distributed" (i.e., Gaussian) wave-packet in real space:

$$f(x) = \frac{1}{\sigma} \exp\left(-\frac{x^2}{2\sigma}\right)$$

for the packet at the origin. The FWHM is gotten by:

$$\frac{1}{2} = \frac{1}{\sigma} \exp\left(-\frac{x_0^2}{2\sigma^2}\right)$$

$$\frac{\sigma}{2} = \exp\left(-\frac{x_0^2}{2\sigma^2}\right)$$

$$\ln\sigma - \ln 2 = -\frac{x_0^2}{2\sigma^2}$$

assume a very narrow wave packet ($\ln\sigma \ll \ln 2$)

$$-\ln 2 = \frac{-x_0^2}{2\sigma^2}\ ; \quad \boxed{x_0 = \sigma\left(\sqrt{2\ln 2}\right)}$$

Now, let's fourier transform the packet to get its k-space representation:

$$I = \mathcal{F}\{f\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma} \exp\left(-\frac{x_0^2}{2\sigma^2}\right) \exp(-2\pi i k x)\, dx$$

$$I = \exp\left(-2k^2\pi^2\sigma^2\right)^{\dagger} = f(k)$$

The FWHM in k-space is

$$\frac{1}{2} = \exp\left(-2k_0^2\pi^2\sigma^2\right)$$

$^{\dagger}$Remember $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$ !

or:

$$\ln(2) = 2 k_0^2 \pi^2 \sigma^2$$

$$k_0 = \frac{1}{\sigma}\left(\frac{1}{\sqrt{2}\pi}\sqrt{\ln(2)}\right)$$

$$x_0 k_0 = \Delta x \Delta k = \left[\cancel{\sigma}\sqrt{2\ln 2}\right]\left\{\frac{1}{\sqrt{2}\pi\cancel{\sigma}}\sqrt{\ln 2}\right]$$

$$\underbrace{\boxed{\Delta x \Delta k = \left(\frac{\ln(2)}{\pi}\right)}}\simeq 1/4$$

a pure <u>number</u> (a constant).