1.

Two lossy dielectric media with permittivities and conductivities \((\varepsilon_1, \sigma_1)\) and \((\varepsilon_2, \sigma_2)\) are in contact. An electric field with a magnitude \(E_1\) is present in medium 1 and makes an angle \(\alpha_1\) with the normal to the interface, as shown in the figure.

a. (3 pts) What are the boundary conditions on \(D\) and \(J\).

b. (3 pts) Find the magnitude and direction of \(E_2\) in medium 2 in terms of \(\alpha_1\), \(E_1\), \(\varepsilon_1\), \(\sigma_1\), and \(\sigma_2\).

2. Consider a radially expanding conducting loop in a spatially varying magnetic field. The magnetic field is given by:

\[
\vec{B} = k \ r \ \vec{a}_z
\]

where \(k\) is a constant.

The loop radius expands at a constant speed \(v\) according to:

\[
r = vt
\]
a. (4 pts) Calculate the induced emf that would be measured between points A and B at time t. Clearly indicate which terminal is (+) and which terminal is (-).

b. (2 pts) Using Lenz law, derive the direction of the current flow if a resistor R is inserted between the A and B terminals. Explain your answer.

3. Assume a dielectric constant $\varepsilon$, a permeability $\mu$, and a conductivity $\sigma$. Also, assume a time-harmonic electric and magnetic field inside a semi-infinite conductor located in a half-plane starting at $z = 0$ and extending to $z = \infty$.

a. (2 pts) From Maxwell’s equations, derive the following homogeneous Helmholtz’s equation which describes the propagation of the electric field (phasor) inside a conductor:

$$\nabla^2 \vec{E} + k_c^2 \vec{E} = 0,$$

where $k_c$ is the propagation wavevector in a conductor and

$$k_c = \omega \sqrt{\mu \varepsilon_c} \quad \text{and}$$

$$\varepsilon_c = \varepsilon \left(1 + \frac{\sigma}{j\omega \varepsilon} \right)$$

is the complex dielectric constant.

You might need the following identity:

$$\nabla \times \nabla \times \vec{E} = \nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E}$$

b. (2 pts) Take $\vec{E}(z) = a_z E_0 e^{-j k_c z}$. Assume a good conductor $\left(\frac{\sigma}{\omega \varepsilon} \gg 1\right)$ and write an expression for the electric field and explicitly show that the electric field in the metal decays by a factor “e” over a distance $\delta = \frac{2}{\mu \sigma \omega}$.

c. (2 pts) Derive an expression for the intrinsic impedance of a good conductor in terms of $\omega$, $\mu$, and $\sigma$.

d. (2 pts) Calculate the dissipated power in the conductor in terms of the conductivity $\sigma$, the angular frequency $\omega$, the permeability $\mu$, and the electric field $E_0$ at the inside surface at $z = 0$ of the good conductor.
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\[
\begin{align*}
D_1n - D_2n &= \rho_s \\
\frac{D_{1n}}{\varepsilon_i} &= \frac{D_{2n}}{\varepsilon_2} \\
J_{1n} &= J_{2n} \\
\frac{J_{1n}}{\sigma_1} &= \frac{J_{2n}}{\sigma_2} \\
E_1 = E_2 \Rightarrow E_2 \sin \alpha_2 &= E_1 \sin \alpha_1 \\
J_1 = J_2 \Rightarrow \sigma_1 E_{1n} &= \sigma_2 E_{2n} \Rightarrow \sigma_1 E_1 \cos \alpha_1 = \sigma_2 E_2 \cos \alpha_2 \\
E_2 &= \sqrt{E_{2n}^2 + E_{2n}^2} = \sqrt{(E_2 \sin \alpha_2)^2 + (E_2 \cos \alpha_2)^2} \\
\therefore E_2 &= E_1 \sqrt{\sin^2 \alpha_1 + \left(\frac{\sigma_1 \cos \alpha_1}{\sigma_2}\right)^2} \\
\tan \alpha_2 &= \frac{\sigma_2}{\sigma_1} \tan \alpha_1 \Rightarrow \alpha_2 = \tan^{-1} \left(\frac{\sigma_2}{\sigma_1} \tan \alpha_1\right) 
\end{align*}
\]
a.

\[ \phi(t) = \int_a^b \bar{B} \cdot d\bar{s} = \int_a^b kr(t) \times 2\pi r(t) \, dr = \int_0^t kvt \times 2\pi vt \, vdt = 2\pi kv^3 \left( \frac{t^3}{3} \right) \]

\[ \phi(t) = 2\pi kv^3 \frac{t^3}{3} \]

\[ \varepsilon = -\frac{d\phi}{dt} = -2\pi kv^3 \frac{t^2}{3} . \]

The B terminal is positive and the A terminal is negative.

b.

The current is flowing clockwise in the loop to oppose the increase of the flux going through the loop.

3.

a.

\[ \nabla \times \vec{E} = -\frac{\partial B}{\partial t} = -j\omega \mu H \]

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = (\sigma + j\omega \epsilon) \vec{E} = j\omega \left( \varepsilon + \frac{\sigma}{j\omega} \right) \vec{E} \equiv j\omega \epsilon \vec{E} \]

where \( \epsilon_e \equiv \varepsilon - j \frac{\sigma}{\omega} = \varepsilon \left(1 - \frac{j\sigma}{\omega \epsilon}\right) \)

Note \( \frac{\sigma}{\omega \epsilon} \div 1 \) for a good conductor, and \( \epsilon_e \approx -\frac{j\sigma}{\omega} \).

Taking the rotational of 1st equation:

\[ \nabla \nabla \times \vec{E} = \nabla \left( \nabla \times \vec{E} \right) = -\nabla^2 \vec{E} = -j\omega \mu \nabla \times \vec{H} = -j\omega \mu \left( \sigma + j\omega \epsilon \right) \vec{E} \]

\[ \nabla^2 \vec{E} = -\omega^2 \mu \epsilon \left(1 + \frac{\sigma}{j\omega \epsilon}\right) \vec{E} = -\omega^2 \mu \epsilon \vec{E} \]

\[ \nabla^2 \vec{E} + k_e^2 \vec{E} = 0 , \text{ where } k_e \equiv \omega \sqrt{\mu \epsilon} . \]

b.

For a good conductor, we have:
$k_e \equiv \omega \sqrt{\mu e_c} \approx \omega \sqrt{\frac{\mu \sigma}{j \omega}} = \sqrt{-j \mu \sigma \omega} = \frac{1-j}{2} \sqrt{\mu \sigma \omega}$

$\vec{E}(z) = \vec{a}_e E_0 e^{-jkz} = \vec{a}_e E_0 e^{-i\sqrt{\frac{\mu \sigma}{2}} z} e^{-i\sqrt{\frac{\mu \sigma}{2}}} z$

Taking $\delta$ as the distance over which the electric field is attenuated by a quantity "e". This distance is called the skin depth.

$$\delta = \sqrt{\frac{2}{\mu \sigma \omega}}$$

c. The characteristic impedance is given by:

$$\eta_e = \sqrt{\frac{\mu}{\epsilon_c}} \approx \sqrt{\frac{j \omega \mu}{\sigma}} = (1+j) \frac{1}{\delta \sigma} \text{ and } \frac{E}{H} = \eta_e$$


d. The dissipated power is given by the Poynting vector:

$$\mathcal{S} = \frac{1}{2} \text{Re}(\vec{E} \times \vec{H}) = \frac{1}{2} \text{Re} \left( \frac{\sigma}{\sqrt{2}} \sqrt{\frac{\sigma}{\omega \mu}} E_0^2 \right) \tilde{z}. \text{ The energy is dissipated in the conductor.}$$

$$|\mathcal{S}| = \frac{1}{2} \sqrt{\frac{\sigma}{2 \omega \mu}} E_0^2$$