Important: In all cases, you must show your work to receive any credit.

(i) (6 pts.)
Consider two random events \( A \) and \( B \) for which
\[
P[A] = \frac{1}{3}, \quad P[B] = \frac{3}{5} \quad \text{and} \quad P[A \cup B^c] = \frac{3}{5}
\]
(\text{where } B^c \text{ indicates the complement of } B.)
Based on the information given, determine whether \( A \) and \( B \) are independent. You must show all work and reasoning to receive credit. Credit will not be given for an unexplained answer, even if it happens to be correct.

(ii) (7 pts.)
A coin has a probability of \( p = 1/3 \) of coming up heads. Imagine you flip the coin repeatedly until you have seen both heads and tails at least once. (I.e., you successively flip until you have seen both sides of the coin.)
(a) What is the probability that the final flip was heads?
(b) Given that it took 4 flips to see both sides of the coin, what is the probability that the final flip was heads?
(c) Given that the final flip was heads, what is the probability that it took 4 flips to see both sides of the coin?

(iii) (7 pts.)
Suppose \( X \) is a zero-mean random variable with variance 1, i.e.,
\[
E[X] = 0, \quad \text{var}[X] = 1
\]
and \( Y \) is a continuous random variable that is related to \( X \) by the following conditional probability density function:
\[
f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}.
\]
(a) Find \( E[Y] \).
(b) Find \( \text{cov}[X, Y] \).
**Important:** In all cases, you must show your work to receive any credit.

(i) **(6 pts.)**

Consider two random events $A$ and $B$ for which

$$P[A] = \frac{1}{3}, \quad P[B] = \frac{3}{5} \quad \text{and} \quad P[A \cup B^c] = \frac{3}{5}$$

(where $B^c$ indicates the complement of $B$.)

Based on the information given, determine whether $A$ and $B$ are independent. You must show all work and reasoning to receive credit. Credit will not be given for an unexplained answer, even if it happens to be correct.

**SOLUTION:**

To see whether $A$ and $B$ are independent, we need to determine whether $P[A \cap B] = P[A]P[B]$.

Using De-Morgan’s law:

$$A^c \cap B = (A \cup B^c)^c$$

and therefore

$$P[A^c \cap B] = 1 - P[A \cup B^c] = 1 - \frac{3}{5} = \frac{2}{5}$$

Next, we observe that

$$B = (A \cap B) \cup (A^c \cap B)$$

and because the events $(A \cap B)$ and $(A^c \cap B)$ are mutually exclusive, we may add the probabilities:

$$P[B] = P[A \cap B] + P[A^c \cap B]$$

This allows us to solve for $P[A \cap B]$:

$$P[A \cap B] = P[B] - P[A^c \cap B] = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$$

Finally, we see that $P[A \cap B] = P[A]P[B]$, which means that $A$ and $B$ are independent.
(ii) (7 pts.)

A coin has a probability of $p = 1/3$ of coming up heads. Imagine you flip the coin repeatedly until you have seen both heads and tails at least once. (I.e., you successively flip until you have seen both sides of the coin.)

(a) What is the probability that the final flip was heads?

(b) Given that it took 4 flips to see both sides of the coin, what is the probability that the final flip was heads?

(c) Given that the final flip was heads, what is the probability that it took 4 flips to see both sides of the coin?

SOLUTION:

(a) We recognize that if the final flip was heads, then the first flip must have been tails, and vice versa. Therefore, to find the probability that the final flip was heads, we need only consider the probability that first flip is tails:

$$P[T] = (1 - p) = \frac{2}{3}$$

(b) We are given that it took $N = 4$ flips to see both sides of the coin. There are only two possible sequences that satisfy this criterion: HHHT and TTTH. We need to calculate the relative probabilities of these two outcomes, knowing that one of them occurred. We can easily evaluate:

$$P[HHHT] = \left(\frac{1}{3}\right)^3 \frac{2}{3} = \frac{2}{81}, \quad P[TTTH] = \left(\frac{2}{3}\right)^3 \frac{1}{3} = \frac{8}{81}$$

The probability that the experiment ends after $N = 4$ trials is therefore

$$P[N = 4] = P[HHHT \cup TTTH] = P[HHHT] + P[TTTH] = \frac{10}{81}$$

where we have made use of the fact that the outcomes HHHT and TTTH are mutually exclusive.

The conditional probability of observing TTTH, given that $N = 4$ is then found to be

$$P[TTTH|N = 4] = \frac{P[TTTH \cap (N = 4)]}{P[N = 4]} = \frac{P[TTTH]}{P[N = 4]} = \frac{4}{5}$$

where we have made use of the property that the outcome TTTH is a subset of the outcome that $(N = 4)$.

(c) Here we apply Bayes’ rule:

$$P[N = 4|\text{final heads}] = P[\text{final heads}|N = 4] \frac{P[N = 4]}{P[\text{final heads}]}$$

In (a), we determined that $P[\text{final heads}] = 2/3$. In (b), we calculated that $P[N = 4] = 10/81$ and $P[\text{final heads}|N = 4] = 4/5$. Using these results, we find

$$P[N = 4|\text{final heads}] = \frac{4}{27}$$
(iii) (7 pts.)

Suppose $X$ is a zero-mean random variable with variance 1, i.e.,

$$E[X] = 0, \quad \text{var}[X] = 1$$

and $Y$ is a continuous random variable that is related to $X$ by the following conditional probability density function:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}.$$

(a) Find $E[Y]$.

(b) Find $\text{cov}[X,Y]$.

SOLUTION:

(a) From the PDF, we see that when $X$ is given, then $Y$ is normally distributed about a mean of $X$. Therefore,

$$E[Y|X] = X$$

Using the law of iterated expectations, we determine

$$E[Y] = E[E[Y|X]] = E[X] = 0$$

(b) The covariance is given by


where we have made use of the fact that $E[X] = 0$, as given. Again, we can use iterated expectation to find $E[XY]$:

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]] = E[X^2]$$

Given the mean and variance of $X$, we can easily conclude that

$$\text{var}[X] = E[X^2] - E[X]^2 = E[X^2] = 1$$

Therefore, we find

$$\text{cov}[X,Y] = 1$$