Part (i) (6 pts.)
Using induction, or otherwise, show that
\[ \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \]

Part (ii) (7 pts.)
The real-valued function \( f(t) \) satisfies the differential equation
\[ f''(t) + 4f(t) = 0 \]
and is such that \( f(0) = 3 \) and \( f'(0) = -8 \). Determine \( \max \{ f(t) : t \geq 0 \} \).

Part (iii) (7 pts.)
Consider a plane \( P_t \) in \( \mathbb{R}^3 \) whose position changes with time. At time \( t \), \( P_t \) is given by
\[ 8x + 6y + z = 50 - t \]
If \( S \) is the surface given by the equation
\[ z = 4 - (x - 3)^2 - 3(y + 1)^2 , \]
determine the value of \( t \) such that \( P_t \) is tangent to \( S \). What are the coordinates of the point of tangency?
Part (i)
The stated identity clearly holds for \( n = 1 \). If it is true for a given value of \( n \) (as shown), then
\[
\sum_{k=1}^{n+1} k^3 = (n+1)^3 + \sum_{k=1}^{n} k^3 \\
= (n+1)^3 + \frac{n^2(n+1)^2}{4} \\
= \frac{(n+1)^2(4n+4+n^2)}{4} \\
= \frac{(n+1)^2(n+2)^2}{4}
\]
i.e., it is true for \( n + 1 \) as well. The stated identity therefore holds for all \( n \).

Part (ii)
The second-order homogeneous linear constant-coefficient differential equation
\[
f''(t) + 4f(t) = 0
\]
has solution
\[
f(t) = Ae^{s_1 t} + Be^{s_2 t}
\]
where \( s_1 \) and \( s_2 \) are the (distinct) roots of the characteristic equation
\[
s^2 + 4 = 0
\]
Since \( s_1 = 2j \) and \( s_2 = -2j \), we have equivalently
\[
f(t) = C \cos(2t + \phi)
\]
where \( C > 0 \) is also the maximum of \( f(t) \). The given initial conditions imply
\[
C \cos \phi = 3 \quad \text{and} \quad -2C \sin \phi = -8
\]
Thus
\[
\max\{f(t) : t \geq 0\} = C = \sqrt{3^2 + 4^2} = 5
\]
Part (iii)

$S$ is a concave surface, an inverted bowl with vertical axis through $(x, y) = (3, -1)$ and with elliptical horizontal cross-sections. To determine the value of $t$ for which $P_t$ is tangent to $S$, we combine the two equations to eliminate $z$:

\[ 50 - t - 8x - 6y = 4 - (x - 3)^2 - 3(y + 1)^2, \]

same as

\[ x^2 - 14x + 3y^2 = -58 + t \]

Completing the square on the left-hand side, we obtain

\[ (x - 7)^2 + 3y^2 = -9 + t \]

This has a real solution $(x, y)$ for $t \geq 9$, which is unique (implying tangency) for $t = 9$. Then $x = 7$, $y = 0$ and $z = -15$.

Alternatively: If $f(x, y) = 4 - (x - 3)^2 - 3(y + 1)^2$, then the tangent plane to $S$ at the point $(x_0, y_0, f(x_0, y_0))$ is given by the equation

\[ z = a_x(x - x_0) + a_y(y - y_0) + f(x_0, y_0) \]

where $a_x = \frac{\partial f}{\partial x}$ and $a_y = \frac{\partial f}{\partial y}$, both evaluated at $(x, y) = (x_0, y_0)$. Thus for suitable $t$, $P_t$ is a tangent plane provided

\[ a_x = -2(x_0 - 3) = -8 \quad \text{and} \quad a_y = -6(y_0 + 1) = -6 \]

Therefore $x_0 = 7$, $y_0 = 0$, $z = f(x_0, y_0) = -15$ and $t = 9$. 