1. (7 pts) A constant voltage $V_0$ is applied to a partially filled parallel-plate capacitor shown below. The permittivity of the dielectric is $\varepsilon$, and the area of the plates is $S$.

   ![Capacitor Diagram]

   a. (3 pts) Evaluate the capacitance $C$ for this parallel plate capacitor.

   b. (4 pts) Find the force on the upper plate. Is the force attractive or repulsive (please justify your answer)?

2. (7 pts) Consider one coil of radius $b_1$ and $N_1$ turns located at the origin. Consider a second coil of radius $b_2$ and $N_2$ turns located a distance $d$ along the z-axis, as shown in the figure below. Assume $d >> b_1, b_2$.

   ![Coil Diagram]
a. (3 pts) Give an expression for the magnetic field at the center of coil 2 due to a current $I_1$ in coil 1. Give your answer in terms of the given parameters.

b. (2 pts) Evaluate the mutual flux $\Phi_{12}$ (flux in coil 2 due to current in coil 1).

c. (2 pts) Evaluate the mutual inductance $L_{12}$.

3. (6 pts) A 50 Ω lossless transmission line is terminated by a load impedance $Z_L = 50 - j75$ Ω. If the incident power is 100 mW,

a. (2 pts) Find the reflection coefficient. Evaluate the tangent of the angle for the voltage reflection coefficient.

b. (2 pts) Evaluate the power dissipated by the load.

c. (2 pts) Evaluate the voltage standing wave ratio on the transmission line.
Qualifying Exam E81, Fall 2013

Solution

#1

a) \[ C_{d_1} = \frac{\varepsilon S}{d_1}; \quad C_d = \frac{\varepsilon_0 S}{y-d_1}. \]

Two conductors in air:

\[ C = \frac{C_d C_{d_1}}{C_d + C_{d_1}} = \frac{\varepsilon \varepsilon_0 S}{\varepsilon (y-d_1) + \varepsilon_0 d_1}. \]

b) \[ \frac{\vec{E}}{y} = \frac{\vec{a}_y}{y} \frac{2 V_0}{\delta y} \quad \text{or} \quad \frac{2}{\delta y} \left( \frac{1}{2} C V_0^2 \right) = \frac{\vec{a}_y}{y} \frac{V_0^2}{2} \frac{dc}{dy} \]

\[ \Rightarrow \frac{\vec{E}}{y} = -\vec{a}_y \frac{\varepsilon^2 \varepsilon_0 S V_0}{2 \left[ \varepsilon (y-d_1) + \varepsilon_0 d_1 \right]^2} \quad \text{(attractive force)} \]

#2

a) We apply Biot-Savart law. Let \( \vec{m} \) be a vector from the source element at \( (d \hat{\theta}, \hat{\phi}) \):

\[ \vec{m} = \vec{a}_b \theta \ l \ d \Phi \]

\[ \vec{r} = \vec{a}_y \ y - \vec{a}_b \ l \]

\[ |\vec{r}| = (a_1^2 + l^2)^{1/2} \]
\[ dl'v'l = \hat{\gamma}_0 l' d\phi' \times (\hat{\gamma}_2 l - \hat{\gamma}_n l) = \hat{\gamma}_0 l' d\phi' + \hat{\gamma}_2 l^2 d\phi' \]

By symmetry, the \( \hat{\gamma}_n \) component cancels.

\[ \vec{B}_{12} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{d^2l' \cdot d\phi'}{(d^2 + l'^2)^{3/2}} = \frac{\hat{\gamma}_2 I_1 I_2 l^2}{2(d^2 + l^2)^{3/2}} \]

b) \[ \phi_{12} = \int \vec{B}_{12} \cdot d\vec{a} \]

If the second loop radius \( l_2 \) is small, then \( \vec{B}_{12} \) is uniform over the area of the second loop. Hence,

\[ \phi_{12} = \frac{\mu_0 I_1 I_2 l^2 l_2^2 \pi}{2(d^2 + l^2)^{3/2}} = \frac{\rho_0 I_1 I_2 l^2 l_2^2 \pi}{2a^3} \]

assuming \( a > l_1, l_2 \)

c) \[ L_{12} = \frac{\rho_1 k_{12}}{I_1} = \frac{\rho_0 I_1 I_2 l^2 l_2^2}{2a^3} \tag{4} \]

#3 a) Reflective Coefficient \( \Gamma \):

\[ \Gamma = \frac{Z_t - Z_0}{Z_t + Z_0} = \frac{50 - 0.75 - 50}{50 - 0.75 + 50} = 0.36 - 0.48 \]

\[ = 0.60 e^{-20\Phi_3} \]

where \( \tan \theta = 0.48 \Rightarrow \theta = 0.937 \text{ rad} \)

b) The transmitted power to the load is:

\[ P_t = (1 - |\Gamma|^2) P_i = (1 - 0.36^2) \times 100 \text{ mW} = 6.4 \text{ mW} \]

c) The standing wave ratio is given by:

\[ \Delta = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1.60}{0.40} = 4 \]