LINEAR SYSTEMS AND SIGNALS — Ph.D. Qualifying Exam Spring 2013

(i) (7 pts.)
Consider a system with input $x(t)$ and output $y(t)$ related as:

$$y(t) + \frac{d}{dt}y(t) = x(t) - \frac{d}{dt}x(t)$$

Find the output of this system when the input is:

$$x(t) = \delta(t) + e^{-t}u(t)$$

(ii) (8 pts.)
Let $r(x)$ be the unit rectangular function defined as:

$$r(x) = \begin{cases} 
1, & -\frac{1}{2} \leq x \leq \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}$$

(a) (2 pts.) Does there exist a linear time invariant (LTI) system which gives the output $r(t/2)$ when the input is $r(t)$, where $t$ is time? If yes, please specify one; if not, please explain why not.

(b) (2 pts.) Does there exist a non-LTI system which gives the output $r(t/2)$ when the input is $r(t)$, where $t$ is time? If yes, please specify one; if not, please explain why not.

(c) (2 pts.) Does there exist an LTI system which gives an output whose Fourier transform is $r(f/2)$ when the input is a signal whose Fourier transform is $r(f)$, where $f$ is frequency? If yes, please specify one; if not, please explain why not.

(d) (2 pts.) Does there exist a non-LTI system which gives an output whose Fourier transform is $r(f/2)$ when the input is a signal whose Fourier transform is $r(f)$, where $f$ is frequency? If yes, please specify one; if not, please explain why not.

(iii) (5 pts.)
Consider a system with input $x[n]$ and output $y[n]$ related as:

$$y[n - 2] - y[n - 1] - 2y[n] = x[n]$$

Is this system bounded-input bounded-output (BIBO) stable? Please justify your answer.
(i)

Note:

\[ H(jw) = \frac{1 - jw}{1 + jw} \quad \text{and} \quad X(jw) = 1 + \frac{1}{1 + jw} = \frac{2 + jw}{1 + jw} \]

Then,

\[ Y(jw) = \frac{(1 - jw)(2 + jw)}{(1 + jw)^2} = \frac{1}{1 + jw} + \frac{2}{(1 + jw)^2} - 1 \]

and

\[ y(t) = e^{-t}u(t) + 2te^{-t}u(t) - \delta(t) \]

(ii)

(a) Yes. \( y(t) = x(t - \frac{1}{2}) + x(t + \frac{1}{2}) \).

(b) Yes. \( y(t) = x^2(t - \frac{1}{2}) + x^2(t + \frac{1}{2}) \). Alternatively, \( y(t) = x(t/2) \).

(c) No.

(d) Yes. \( y(t) = 2x(2t) \).

(iii)

The impulse response of this system is:

\[ h[n] = -\frac{1}{3} \left( (-1)^n + \left(\frac{1}{2}\right)^{n+1} \right) u[n] \]

Then, lower bounding the sum of absolute values of the impulse response by a sum over the even numbers only:

\[ \sum_{n=0}^{\infty} |h[n]| > \sum_{n=0}^{\infty} |h[2n]| = \alpha \sum_{n=0}^{\infty} \left( 1 + \left(\frac{1}{2}\right)^{2n+1} \right) = \infty \]

Therefore, the system is not BIBO stable.