Part (a) (7 pts.)
A bag contains three dice with identical appearance.

- One die is fair (each outcome has probability 1/6).
- The other two are biased with \( p_1 = p_2 = p_3 = (1 - \alpha)/6 \) and \( p_4 = p_5 = p_6 = (1 + \alpha)/6 \), where \( 0 < \alpha < 1 \).

One die is drawn at random from the bag and rolled twice.

(a1) (5 pts.) Determine the probability of \( A = \) “the maximum of the two scores (rolls) is 4”.

(a2) (2 pts.) Given that \( A \) occurs, what is the conditional probability that the chosen die was fair?

Part (b) (6 pts.)
Let \( X \) be an exponentially distributed random variable with density

\[
f_X(x) = 3e^{-3x}, \quad x \geq 0
\]

and \( f_X(x) = 0 \) for \( x < 0 \). If for all \( x > 0 \),

\[
P[Y = 2 - 3x \mid X = x] = P[Y = 2 + 3x \mid X = x] = 1/2,
\]

determine the probability of the event

\[
\{0 \leq X \leq 1, \ 0 \leq Y \leq 1\}
\]

Part (c) (7 pts.)
Let the sample space \( \Omega \) consist of all points in the interior of an equilateral triangle, each side of which has length 2. If the random outcome \( \omega \) has uniform distribution over \( \Omega \),

(c1) (5 pts.) determine the cumulative distribution function of \( X = \) distance of \( \omega \) from the nearest side; and

(c2) (2 pts.) determine the expectation \( E[X] \).
Solutions - Probability (F2014 Qual)

(a1)

\[ P[Fair] = \frac{1}{3}, \quad P[Biased] = \frac{2}{3} \]
\[ P[A|F] = \frac{1}{6} \times \frac{1}{6} + 2 \times \frac{1}{6} \times \frac{1}{2} = \frac{7}{36} \]
\[ P[A|B] = \frac{(1+a)(1+a)}{36} + 2 \times \frac{(1+a)(1-a)}{12} \]

\[ = \frac{7}{36} + \frac{a}{27} - \frac{(5/54)a^2}{2} \]

(a2)

\[ P[Fair|A] = \frac{P[Fair,A]}{P[A]} \]
\[ = \frac{7}{21 + 4a - 10a^2} \]

(b)

The range of \((X,Y)\) consists of two half-lines,
\[ \{(x,y): x>0, y = 2-3x\} \text{ and } \{(x,y): x>0, y = 2+3x\} \]
Only the first component has nonempty intersection
A with the event of interest, given by
\[ A = \{(x,y): x \in [1/3,2/3], y = 2-3x\} \]

Thus
\[ P[X \in [0,1], Y \in [0,1]] = P[(X,Y) \in A] \]

which is the integral over \([1/3,2/3]\) of
\[ f(x)P[(X,Y) \in A | X = x] = (1/2)f(x), \]
(with f(x) = pdf of X at x). The result is

\[
(1/2) * P[X \in [1/3, 2/3]] = (\exp(-1) - \exp(-2))/2
\]

(c1)

X takes values between 0 and \( \sqrt{3}/3 \). For x in that interval,

\[
F(x) = 1 - P[X > x] = 1 - \frac{\text{Area}(T(x))}{\text{Area}(\Omega)}
\]

where T(x) is also an equilateral triangle with

\[
\text{Height}(T(x)) = \sqrt{3} - x - 2x = \sqrt{3} - 3x
\]

Thus over the range of X,

\[
F(x) = 1 - (1 - \sqrt{3}x)^2 = 2\sqrt{3}x - 3x^2
\]

(c2)

Can differentiate to obtain f(x), or recall that for a nonnegative variable X, \( E[X] = \int 1 - F(x) \) over the range of X.

\[
E[X] = \int (1-\sqrt{3}x)^2 \text{ over } [0, \sqrt{3}/3]
\]

\[
= \sqrt{3}/9
\]