Part 1 (7 pts.)
Consider a $M \times M$ matrix. Each element of the matrix is either ♦ with probability $p$ or ★ (with probability $q = 1 - p$), independently of all other elements.

(1a) (3 pts.) If $A$ is the event that each row of the matrix contains at most one ♦, compute $P[A]$.

(1b) (4 pts.) If $B$ is the event that each row and each column of the matrix contains exactly one ♦, compute $P[B]$.

Part 2 (6 pts.)
Let $X$ be a random integer with probability mass function
\[(\forall x \in \mathbb{Z}) \quad P[X = x] = \frac{1}{3} \cdot 2^{-|x|}\]
Let $Y$ be a noisy version of $X$; specifically,
\[Y = X + V,\]
where $V$ is a random variable independent of $X$, having a triangular density:
\[f_V(v) = \begin{cases} \frac{2 - |v|}{4}, & |v| \leq 2 \\ 0, & |v| > 2 \end{cases}\]
Evaluate the conditional probability $P[X < 0 \mid Y = 0]$.

Part 3 (7 pts.)
Consider a circle of unit radius centered at point $C$. Let $A$ be a fixed point on the circumference of the circle and $B$ be a random point with a uniform distribution over the circumference.
Let $S$ be the area of the triangle $ABC$.

(3a) (1 pt.) Determine the maximum possible value of $S$.

(3b) (4 pts.) Determine the expectation $E[S]$.

(3c) (2 pts.) Does the answer to (3b) change if $A$ is also random, independent of $B$, and uniformly distributed over the circumference? Explain your answer.
Probability Fall 2015 - Solution

Part 1

1a) Each row has $M$ elements which are independently diamond ("1") or asterisk ("0"). Thus for a given row,

$$P[\text{at most one } "1"] = q^M + M*p*q^{M-1}$$

Since there are $M$ independently generated rows,

$$P[\text{each row contains at most one } "1"] = ( q^M + M*p*q^{M-1} )^M$$

1b) There are $M!$ configurations where each row and each column contains exactly one "1". Therefore

$$P[\text{each row and each column contains exactly one } "1"] = (M!)*(p*q^{M-1})^M$$

Part 2

Since $Y$ has a continuous distribution, the conditional probability

$$P[X<0 \mid Y=0]$$

is the limit, as $d \rightarrow 0$, of

$$P[X<0 \mid Y \in D]$$,

where $D$ is the event $\{ 0 < Y < d \}$. With $f(.)$ denoting the density of $Y$, we have the approximations

$$P[X<0, Y \in D] \sim \sum \{ P[X=x]*f(-x)*d , x<0 \}$$

and

$$P[Y \in D] \sim \sum \{ P[X=x]*f(-x)*d , \text{ all } x \}$$

Dividing and taking the limit as $d \rightarrow 0$, we have

$$P[X<0 \mid Y=0] = \frac{\sum \{ P[X=x]*f(-x) , x<0 \}}{\sum \{ P[X=x]*f(-x)*d , \text{ all } x \}}$$

With $f(.)$ as given, this reduces to $a/b$, where
\[ a = P[X=-1]*f(1) = (1/3)*(1/2)*(1/4) = 1/24 \]

and
\[ b = P[X=-1]*f(1) + P[X=0]*f(0) + P[X=1]*f(-1) = 1/4 \]

Thus \( P[X<0 \mid Y=0] = 1/6 \)

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**Part 3**

If \( T \) (between \(-\pi\) and \( \pi\)) is the random angle corresponding to the arc \( AB \), then

- the conditional distribution of \( T \) given \( A \) is uniform over \((-\pi, \pi]\) and so is the (unconditional) distribution of \( T \);
- the area \( S \) of the triangle \( ABC \) equals \((1/2)*\sin(|T|)\)

**3a)** Maximum value of \( S \) is \( 1/2 \)

**3b)** \( E[S] = \text{integral of } \sin(t)/(2\pi) \text{ over } [0,\pi] = 1/\pi \)

**3c)** If \( A \) is independent of \( B \), then the conditional distribution of \( B \) given a particular position \( Ao \) of \( A \) is the same as the unconditional (uniform) distribution of \( B \); thus

\[ E[S|A=Ao] = 1/\pi \]

by the calculation above. Taking an expectation with respect to the distribution of \( A \) (which doesn't need to be uniform) results again in \( E[S] = 1/\pi \).