Part 1 (6 pts.)
Sequences \(x[n]\) and \(\tilde{x}[n]\) both have finite duration, with nonzero samples as shown below.

\[
\begin{array}{c}
\vdots \quad \vdots \\
0 \quad 6 \quad \vdots \\
x[n] \\
\vdots \quad \vdots \\
0 \quad 3 \quad 11 \quad 13 \\
\vdots \\
\end{array}
\quad \begin{array}{c}
\vdots \quad \vdots \\
0 \quad 6 \quad \vdots \\
\vdots \quad \vdots \\
0 \quad 3 \quad 11 \quad 13 \\
\vdots \\
\end{array}
\]

(1a) (3 pts.) Give an example of a sequence \(v[n]\) consisting entirely of nonzero samples (i.e., \(v[n] \neq 0\) for all \(n\)) such that the convolution of \(v[n]\) and \(x[n]\) (left graph) is the all-zeros sequence:

\[ (\forall n \in \mathbb{Z}) \quad (v * x)[n] = 0 \]

(1b) (3 pts.) Suppose the response of a linear time-invariant system to the input sequence \(x[n]\) is the (output) sequence \(y[n]\). What is the response \(\tilde{y}[n]\) of the same system to the input sequence \(\tilde{x}[n]\) (right graph)? Express \(\tilde{y}[\cdot]\) in terms of \(y[\cdot]\).

Part 2 (7 pts.)
Consider the periodic signal \(x(t)\) of period \(T_0 = 6\) sec, defined (over the time interval \([-3, 3]\)) by

\[
x(t) = \begin{cases} 
1, & |t| \leq 1 \\
0, & 1 < |t| \leq 3
\end{cases}
\]

(2a) (3 pts.) Sketch the general form of the Fourier transform \(X(j\omega)\), where \(\omega\) is in rad/sec. The horizontal \((\omega)\) axis should be marked in detail. You need not provide values for the vertical coordinates.

(2b) (4 pts.) If \(x(t)\) is the input to a linear time-invariant system with frequency response

\[
H(j\omega) = \begin{cases} 
1, & |\omega| \leq \pi/2 \\
0, & |\omega| > \pi/2
\end{cases}
\]

determine the system’s output \(y(t)\).

Part 3 (7 pts.)
Consider the causal linear time-invariant system with transfer function

\[
H(s) = \frac{s - 1}{s^2 + bs + 3},
\]

where \(b\) is real-valued.

(3a) (4 pts.) For which, if any, values of \(b\) does the system’s impulse response \(h(t)\) have both of the following properties:

- it is bounded (over all time \(t\)); and
- there are infinitely many values of \(t\) greater than zero for which \(h(t) = 0\).

(3b) (3 pts.) If \(b = 1\), determine the response (output) \(y(t)\) of the system to the input signal given by \(x(t) = \cos t\), for all \(t \in \mathbb{R}\). Your answer should be real-valued.
Part 1

---

(1a)

If \( z[n] \) is the convolution of \( x[n] \) and \( v[n] \), then

\[
    z[n] = v[n] + \ldots + v[n-6]
\]

This can be made zero for all \( n \) by choosing \( v[n] \) periodic with period 7 samples, and sum = 0 over one period, e.g.,

\[
    v[0 \text{ through } 6] = (1, 2, 3, 4, 5, 6, -21)
\]

(1b)

Using ‘ instead of ~)

\[
    x'[n] = x[n] - x[n-4] + x[n-7]
\]

By linearity and time invariance,

\[
\]

Part 2

---

Period \( T_0 = 6 \text{ sec} \Rightarrow \text{Fundamental } w_0 = 2\pi/6 = \pi/3 \text{ rad/sec} \)

\[
    x(t) = \text{sum of } X[k]\exp(jk\omega_0 t) \text{ over all integers } k
\]

(i.e., \( X[k] \) is the \( k \)th Fourier series coefficient)

(2a)

\[
    \tilde{X}(\tilde{w}) = \text{sum of } 2\pi i X[k]\delta(w-kw_0) \text{ over all } k
\]

Thus the graph consists exclusively of impulses at positions

\[
    w = k\omega_0 = k(\pi/3)
\]

(2b)

\( \mathcal{H}(\tilde{w}) \) is an ideal lowpass filter characteristic with
cutoff frequency \( wc = \pi/2 \) and gain = 1. Harmonics \( k = -1, 0 \) and +1 are within the passband \( |w|<|wc| \).

\[
    X[0] = (1/6) \ast \text{(area under } x(.)) \text{ over }[-3,3]) = 1/3
\]

\[
    X[1] = (1/6) \ast \text{(integral of exp(-jpi*t) over }[-3,3])
        = \sin(\pi/3)/\pi = \sqrt{3}/(2\pi)
\]

Thus

\[
    y(t) = 1/3 + (\sqrt{3}/\pi) \cos(pi*t/3).
\]
Linear Systems Solution

Part 3

(3a)

\[ h(t) = [C_1 \exp(a_1 t) + C_2 \exp(a_2 t)]u(t) \]

where \( a_1 \) and \( a_2 \) are the poles (roots of the quadratic denominator of \( H(s) \)).

\( h(.) \) is bounded iff the real parts of \( a_1 \) and \( a_2 \) are both \( \leq 0 \);
and it has infinitely many zero crossings in positive time
iff the imaginary parts of \( a_1 \) and \( a_2 \) are nonzero. Since

\[ a_1 = (-b + \sqrt{b^2 - 12})/2, \quad (-b - \sqrt{b^2 - 12})/2, \]

the conditions are met iff

\[ b \geq 0 \quad \text{and} \quad b^2 - 12 < 0, \quad \text{i.e.,} \quad b \in [0, 2\sqrt{3}) \]

(3b)

If \( x(t) = \cos(t) \) for all \( t \), then

\[ y(t) = A\cos(t + q), \]

where \( A\exp(jq) = H(jw) \) at \( w = 1 \). Since

\[ H(j1) = (j - 1)/(-1 + j + 3) = -(1/5) + j*(3/5) \]
\[ = (\sqrt{10}/5)*\exp(j*(-\arctan(3)+\pi)) \]

it follows that

\[ y(t) = (\sqrt{10}/5)*\cos(t-\arctan(3)+\pi) \]
\[ (\text{also}) = -(1/5)*\cos(t) - (3/5)*\sin(t) \]