1. Suppose that, to get to work, one can either drive or take the train. When driving, the amount of time (in minutes) it takes is uniformly distributed over the interval $[15, 25]$. When taking the train, the time waiting for the train is exponentially distributed with parameter $1/5$ (that is, with pdf of $\frac{1}{5}e^{-w/5}$ for $w \geq 0$ and 0 otherwise), followed by the train ride which will take 15 minutes (such that the mean overall travel time is the same for each). \(6\) total points

(a) Suppose that this person has an important meeting in 18 minutes, and find the probability of being late with each of the two options.

\textbf{Solution:} \ Let $D$ be the time it takes to drive, $W$ the time waiting for the train, and $T$ the total time with the train.

$$P[D > 18] = \frac{25 - 18}{25 - 15} = \frac{7}{10} \quad P[T > 18] = P[W > 3] = e^{-3/5}$$

(b) Find the probability that taking the train would take longer than driving. \(3\) points

\textbf{Solution:}

$$P[T > D] = \int_{15}^{25} P[T > r] \cdot \frac{1}{10} \, dr$$

$$= \int_{15}^{25} P[W > r - 15] \cdot \frac{1}{10} \, dr$$

$$= \frac{1}{10} \cdot \int_{15}^{25} e^{-\frac{r-15}{5}} \, dr$$

$$= \frac{1}{10} \cdot \frac{5}{10} \left[ e^{-\frac{r-15}{5}} \right]_{r=15}^{25}$$

$$= \frac{1}{2} \left( 1 - e^{-2} \right)$$
2. An urn contains 4 white balls and 6 black balls. A (first) ball is chosen at random. It is then replaced, along with 2 more balls of the same color (such that there are then 12 balls in the urn).

(7 total points)

(a) Another (second) ball is then drawn at random from the urn. Find the probability that this ball is white. 

Solution:

\[
P \left[ B_2 = w \right] = P \left[ B_1 = w \right] P \left[ B_2 = w \mid B_1 = w \right] + P \left[ B_1 = b \right] P \left[ B_2 = w \mid B_1 = b \right]
\]

\[
= \frac{2}{5} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{3}
\]

\[
= \frac{2}{5}
\]

(b) Given that this second ball is white, find the probability that the first ball drawn from the urn was black.

Solution:

\[
P \left[ B_1 = b \mid B_2 = w \right] = \frac{P \left[ B_1 = b, B_2 = w \right]}{P \left[ B_2 = w \right]}
\]

\[
= \frac{\frac{3}{5} \cdot \frac{1}{3}}{\frac{2}{5}} = \frac{1}{2}
\]

(c) The second ball is replaced (such that there are still 12 balls in the urn), and a third ball is drawn. If the second and third balls are both white, find the probability that the first ball was black.

Solution:

\[
P \left[ B_2 = w, B_3 = w \right] = P \left[ B_1 = w \right] P \left[ B_2 = w, B_3 = w \mid B_1 = w \right]
\]

\[
= \frac{2}{5} \cdot \left( \frac{1}{2} \right)^2 + \frac{3}{5} \cdot \left( \frac{1}{3} \right)^2
\]

\[
= \frac{1}{10} + \frac{1}{15} = \frac{1}{6}
\]

\[
P \left[ B_1 = b \mid B_2 = w, B_3 = w \right] = \frac{P \left[ B_1 = b, B_2 = w, B_3 = w \right]}{P \left[ B_2 = w, B_3 = w \right]}
\]

\[
= \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{2}{5}
\]
3. Suppose that a point $Z$ is picked uniformly at random from the perimeter of a unit circle; that is, from a circle of radius 1 with center at the origin, $(0, 0)$. Now let $X$ be the $x$-coordinate of this point $Z$. (7 total points)

(a) Find the distribution and density of $X$. (5 points)

**Solution:** Given $-1 \leq x \leq 1$,

$$F_X(x) = P[X \leq x] = 2 \cdot P[\cos^{-1}(x) \leq \Theta \leq \pi] = \frac{\pi - \cos^{-1}(x)}{\pi}$$

where $\theta = \cos^{-1}(x)$ is given by $\theta \in [0, \pi]$ such that $\cos(\theta) = x$, and so

$$F_X(x) = \begin{cases} 
0 & \text{if } x < -1; \\
\frac{\pi - \cos^{-1}(x)}{\pi} & \text{if } -1 \leq x < 1; \\
1 & \text{if } x \geq 1.
\end{cases}$$

Then,

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 
\frac{1}{\pi \sqrt{1 - x^2}} & \text{if } -1 \leq x \leq 1; \\
0 & \text{otherwise}.
\end{cases}$$

Alternatively, one could find the density first in a few ways, and then integrate to get the distribution.

(b) Let $Y = |X|$. Find the expectation $E[Y]$. (2 points)

**Solution:**

$$E[Y] = 2 \cdot \int_0^1 \frac{x}{\pi \sqrt{1 - x^2}} \, dx = \left. -\frac{2}{\pi} \sqrt{1 - x^2} \right|_{x=0}^{1} = \frac{2}{\pi}$$