(1) (7 pts.) Find the general solution of the following simultaneous differential equations:

(a) \[
\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = y - 2x \quad (3 \text{ pts.})
\]

(b) \[
\frac{d^2x}{dt^2} = x - y, \quad \frac{d^2y}{dt^2} = y - x \quad (4 \text{ pts.})
\]

(2) (6 pts.) Evaluate the integral

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x^2 \, dx \, dy}{1 + \sqrt{x^2 + y^2}}
\]

(3) (7 pts.) Let \( \hat{u} \) be a unit vector on the Cartesian plane. For an arbitrary vector \( v \), let \( w \) be the reflection (mirror image) of \( v \) across the line containing \( \hat{u} \). Show that

\[
w = 2(\hat{u} \cdot v)\hat{u} - v
\]

Writing \( w = Rv \) and taking the Cartesian coordinates of \( \hat{u} \) and \( v \) to be respectively \( (u_1, u_2) \) and \( (v_1, v_2) \), find the components of the matrix \( R \). Verify that \( R^2 = I \), where \( I \) is the identity matrix.
Solutions

(1) (a) From first equation \( y = \frac{x}{2} - \frac{1}{2} \frac{dx}{dt} \). Substitute in second equation \( \frac{d^2x}{dt^2} - 2 \frac{dx}{dt} - 3x = 0 \).

This can be written as \( D^2 - 2D - 3 = (D-3)(D+1) \). \( D = 3,-1 \)

General solution is \( x = Ae^{-t} + Be^{3t} \) and \( y = Ae^{-t} - Be^{3t} \)

(b) From first equation \( y = x - \frac{d^2x}{dt^2} \). Substitute in second equation \( \frac{d^4x}{dt^4} - 2 \frac{d^3x}{dt^3} = 0 \)

\( D^4 - 2D^2 = D^2(D^2 - 2) = 0 \). \( D = \pm \sqrt{2},0 \). General solution is \( x = At + B + Ce^{\sqrt{2}t} + De^{-\sqrt{2}t} \)

\( y = At + B - Ce^{\sqrt{2}t} - De^{-\sqrt{2}t} \)

(2) Substitute \( x = r \cos(\theta), y = r \sin(\theta) \) switch infinite domain of integration to polar coordinates.

Integral becomes \( \int \int_{0}^{\infty} \int_{0}^{\infty} r^2 \cos^2(\theta) dr d\theta \) \( \frac{(1+r)^3}{(1+r)^2} = \int \int_{0}^{\infty} r^3 (1+\cos(2\theta)) dr d\theta \) \( \frac{1}{2(1+r)} \). \( \pi \int_{0}^{\infty} \frac{r^3}{(1+r)^3} dr \)

Substitute \( 1+r = x \). Integral becomes \( \int_{1}^{\infty} (x-1)^3 dx \) \( \frac{1}{x^2} + \frac{3}{x^3} + \frac{3}{x^4} - \frac{1}{x} \). \( \pi \int_{1}^{\infty} \frac{(x-1)^3}{x^3} dx \)

\( \left[ \pi \left( -\frac{1}{x} + \frac{3}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} \right) \right]_{1}^{\infty} = \frac{\pi}{4} \)

(3) \( \mathbf{v} \cdot \hat{u} = |\mathbf{v}| \cos(\theta) \) where \( \theta \) is the angle between the two vectors. Also \( \mathbf{w} \cdot \hat{u} = |\mathbf{w}| \cos(\theta) \). All three vectors are in the same plane.

\( \mathbf{v} + \mathbf{w} = 2|\mathbf{v}| \hat{u} \cos(\theta) \) gives \( \mathbf{w} = 2(\hat{u} \cdot \mathbf{v}) \hat{u} - \mathbf{v} \) Q.E.D.

Substitute \( \hat{u}, \mathbf{v} \) as column vectors \( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) note that \( u_1^2 + u_2^2 = 1 \) gives

\( \mathbf{R} = \begin{pmatrix} u_1^2 - u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_2^2 - u_1^2 \end{pmatrix} \) and it follows that \( \mathbf{R}^2 = \mathbf{I} \) Q.E.D.