Part 1: Consider that a message with $N$ bits is stored in a noisy memory device that flips each bit with probability $p$. Flip events are independent across bits. Let $X$ be the random variable that quantifies the total number of bit flips.

A) (3 points) Assume that $E[X] = 5$ and $N = 10$, where $E[X]$ represents the expectation or mean value of $X$. Determine $P(X = 2)$ and $E[X^2]$.

B) (4 points) Suppose that $N = 10^3$ and that information can be correctly recovered from the memory when at most 2 bits are flipped. An error occurs when 3 or more bits are flipped. The cost of a memory chip is given by $2 \times 10^3 e^{-Np}$ and the cost of making an error is $10^3$. Choose $p$ that minimizes the expected total cost given by $10^3 (P(error) + 2e^{-Np})$. (Hint: You should resort to the Poisson-approximation to solve this problem.)

Part 2: Consider that a bucket contains $N$ components that are labeled from 1 to $N$. Components are taken uniformly randomly from the bucket without replacement. Consider that you have taken $t$ components and placed them in a shopping bin.

A) (4 points) What is the probability that the component with label 1 is in the bin? Give a formula that depends on $t$ and $N$, for $1 \leq t \leq N$.

B) (2 points) What is the probability that all the components in the bin have label less than or equal to $t$? Give a formula that depends on $t$ and $N$, for $1 \leq t \leq N$.

Part 3: Let $W_1$ and $W_2$ be two independent random variables that are uniformly distributed between $-1$ and 1.

A) (4 points) Sketch and determine explicitly $P(W_1 + \alpha \geq W_2)$ as a function of $\alpha$.

B) (3 points) Assume that $Z = \max\{W_1, W_2\}$. Determine the probability density function of $Z$ and compute $E[Z]$. 


SOLUTIONS

SOLUTION 1-A) \( p = 0.5, \ P(X = 2) = 45 \times \frac{1}{1024}, \ E[X^2] = 10 \times 0.5^2 \)

SOLUTION 1-B) The cost can be written as \( 10^3(1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) + 2e^{-\lambda}) \) in terms of \( \lambda = np \). Take derivative to get \( 10^3e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2} - (\lambda + 1) - 2) \). The derivative is zero when \( \lambda = 2 \), or equivalently \( p = 2 \times 10^{-3} \).

SOLUTION 2-A) \( \binom{N - 1}{t - 1} = \frac{t}{N} \)

2-B) \( \frac{1}{\binom{N}{t}} \)

SOLUTION 3-A) \( P(W_1 + \alpha \geq W_2) = \begin{cases} 0 & \alpha \leq -2 \\ \frac{\alpha + 2}{8} & -2 < \alpha \leq 0 \\ \frac{1 - (\alpha - 2)^2}{8} & 0 < \alpha \leq 2 \\ 1 & \alpha > 2 \end{cases} \)

SOLUTION 3-B) \( P(Z \leq z) = P(W_1 \leq z)P(W_2 \leq z) = \begin{cases} 0 & z < -1 \\ 1 & z \geq 1 \\ \frac{1}{4}(z + 1)^2 & \text{otherwise} \end{cases} \)

So, the PDF is the derivative of the CDF:

\( f_Z(z) = \begin{cases} 0 & |z| \geq 1 \\ \frac{1}{2}(z + 1) & \text{otherwise} \end{cases} \)

So, \( E[z] = 0.5 \int_{-1}^{1} z^2 + zdz = 0.5(\frac{1}{3}z^3 + \frac{1}{2}z^2)|_{-1}^{1} = \frac{1}{6}(1 - (-1)) = \frac{1}{3} \)