PhD Qualifying Exam, Spring 2013

Electromagnetism

1. (6 pts) Imagine two spheres of radius R, made up of fixed charges in free space, such that they carry a uniform charge density $+\rho$ and $-\rho$, respectively. The two spheres are placed such that their centers are along the x-axis and in a way that they partially overlap. Take the separation between the centers of the two spheres as $d$ ($d < 2R$).

   a. (4 pts) Find an expression for the total electric field in the region of overlap.
   b. (2 pts) Show that the total electric field in the region of overlap is constant and find its value.

2. (7 pts) Determine the mutual inductance $L_{12}$ between a conducting triangular loop with a base of length $b$ and making an angle of 60° and a very long straight wire, as shown below. You can assume a current $I_1$ flowing in the triangle and a current $I_2$ flowing in the straight wire.

   a. (2 pts) Write the expression for $\vec{B}_2$ that is caused by a current $I_2$ in the long straight wire.
   b. (3 pts) Evaluate the flux linkage $\Lambda_{21}$
   c. (2 pts) Evaluate the mutual inductance $L_{12}$. How different is $L_{21}$ from $L_{12}$?

3. (7 pts) Consider a uniform plane wave incident obliquely on a non-magnetic ($\mu_1 = \mu_2 = \mu_0$) planar dielectric surface with parallel polarization ($\vec{E}_i$ is parallel to the plane of incidence defined by the incident and reflected waves). $\beta_1$ is the propagation wave vector in the incident medium. $\beta_2$ is the propagation wave vector in the dielectric medium where the transmitted wave is propagating. The index of refraction in the incident medium is $n_1$ and the index of refraction in the transmitted
medium is $n_2$. Take $\theta_i$ as the incidence angle measured from the normal to the interface. Take the incident wave as:

$$E_i(x,y) = E_{i0} \left( a_x \cos(\theta_i) - a_z \sin(\theta_i) \right) e^{-j\beta(x \sin \theta_i + z \cos \theta_i)}$$

$$H_i(x,y) = a_y \frac{E_{i0}}{\eta_1} e^{-j\beta(x \sin \theta_i + z \cos \theta_i)}$$

where $\eta$ is the intrinsic impedance.

a. (2 pts) Show that the reflection coefficient for the amplitude of the electric field can be written as:

$$\Gamma_\parallel = \frac{E_{r0}^*}{E_{i0}} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t}$$

where $n$ is the index of refraction.

b. (2 pts) Show that there is a combination of $n_1$, $n_2$, and $\theta$, which will make $\Gamma_\parallel = 0$.

c. (3 pts) The angle where there will be no reflection is called the Brewster angle $\theta_{\parallel \parallel}$. Show that the Brewster angle is given by the following expression.

$$\theta_{\parallel \parallel} = \arctg \left( \frac{n_2}{n_1} \right)$$
Solution for E&M
Spring 2013

# 1
Let us calculate the \( \vec{E} \) field for the case of one sphere for a point anywhere inside the sphere at a distance \( r \).

Using Gauss' law, we have

\[ \int \vec{E} \cdot d\vec{a} = \Phi \]

\[ \varepsilon_0 \vec{E} \cdot 4\pi r^2 = \Phi \]

\[ \Rightarrow \vec{E} = \frac{\Phi}{4\pi \varepsilon_0 r^2} \]

If we take 2 spheres aligned on the \( x \)-axis and using the principle of superposition, the total \( \vec{E} \) field is given by

\[ \vec{E} = \frac{\Phi_1}{4\pi \varepsilon_0} \hat{\mathbf{r}}_1 - \frac{\Phi_2}{4\pi \varepsilon_0} \hat{\mathbf{r}}_2 \]

The component of electric field outside

\[ \Rightarrow \vec{E}_{\text{outside}} = \left( \frac{\Phi_1}{3 \varepsilon_0} \cos \theta_1 + \frac{\Phi_2}{3 \varepsilon_0} \cos \theta_2 \right) \hat{\mathbf{r}} \]

\[ \Rightarrow \vec{E}_{\text{outside}} = \frac{\Phi}{3 \varepsilon_0} \hat{\mathbf{r}} \]

This is independent of position of charges.
# 2  a) Using Ampere's Circuital Law.

\[ \vec{B}_2 = \vec{a}_p \frac{\mu_0 I_2}{2\pi r} \]

b) \[ A_{21} = \oint_S \vec{B}_2 \cdot d\vec{s}, \]

where \( d\vec{s} = \vec{a}_p \theta \, d\theta \)

\[ \sin \theta = \left[ (d+b) - r \right] \tan 60^\circ \]

\[ = \sqrt{3} \left[ (d+b) - r \right] \]

\[ \Lambda_{21} = \sqrt{3} \frac{\mu_0 I_2}{2\pi} \int_0^{2\pi} \frac{1}{r} \left[ (d+b) - r \right] \, d\theta \]

\[ = \sqrt{3} \frac{\mu_0 I_2}{2\pi} \left[ (d+b) \ln \left( \frac{1+b}{d} \right) - b \right] \]

\[ \frac{L_{21}}{I_2} = \frac{\Lambda_{21}}{I_2} = \sqrt{3} \frac{\mu_0}{2\pi} \left[ (d+b) \ln \left( \frac{1+b}{d} \right) - b \right] \]

Let \( L_{21} = L_{12} \) for mutual inductance.
\[ E_i (x, y, z) = E_0 \left( \hat{\alpha}_x \cos \theta_i - \hat{\alpha}_z \sin \theta_i \right) e^{-j \beta (x \sin \theta_i + y \cos \theta_i)} \]

\[ H_i (x, y, z) = \frac{E_0}{n_1} e^{-j \beta (x \sin \theta_i - y \cos \theta_i)} \]

\[ E_n (x, y, z) = E_0 \left( \hat{\alpha}_x \cos \theta_n + \hat{\alpha}_z \sin \theta_n \right) e^{-j \beta (x \sin \theta_n - y \cos \theta_n)} \]

\[ H_n (x, y, z) = -\frac{E_0}{n_2} e^{-j \beta (x \sin \theta_n + y \cos \theta_n)} \]

where \( \eta \) is the intrinsic impedance.

The electromagnetic electric field and magnetic field in medium 2:

\[ E_\parallel (x, y, z) = \frac{E_0}{n_0} \left( \hat{\alpha}_x \cos \theta_\parallel - \hat{\alpha}_z \sin \theta_\parallel \right) e^{-j \beta (x \sin \theta_\parallel + y \cos \theta_\parallel)} \]

\[ H_\parallel (x, y, z) = \frac{E_0}{n_2} e^{-j \beta (x \sin \theta_\parallel + y \cos \theta_\parallel)} \]

Continuity of tangential components of \( E \) and \( H \) at \( z = 0 \)

\[ (E_{\parallel 0} + E_{\parallel i}) \cos \theta_i = E_{\parallel 0} \cos \theta_\parallel \]

\[ \frac{1}{n_1} (E_{\parallel 0} - E_{\parallel i}) = \frac{1}{n_2} E_{\parallel 0} \]

Solving for \( E_{\parallel 0} \) and \( E_{\parallel i} \), we get:

\[ \frac{E_{\parallel i}}{E_{\parallel 0}} = \frac{\eta_2 \cos \theta_\parallel - \eta_1 \cos \theta_i}{\eta_1 \cos \theta_\parallel + \eta_2 \cos \theta_i} \]

where \( \eta = \sqrt{n} \)

\[ = \frac{m_1 \cos \theta_\parallel - m_3 \cos \theta_i}{m_1 \cos \theta_\parallel + m_2 \cos \theta_i} \]
1) Note that \( \Gamma_{\|} = 0 \) if

\[
m_1 \cos \theta_{\|} = m_2 \cos \theta_{\|} \quad \text{where initial } \Theta - \Theta_{\|} \to \text{Breiten.}
\]

Then

\[
\cos \theta_{\|} = \sqrt{1 - \sin^2 \theta_{\|}} = \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \Theta_{\|}} \quad (2)
\]

2) Using (2) in (1):

\[
m_1 \left(1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \Theta_{\|}\right)^{\frac{y_2}{2}} = m_2 \left(1 - \sin^2 \Theta_{\|}\right)^{\frac{y_2}{2}}
\]

\[
m_1^2 - m_2^2 = \left[\frac{\sin^2 \Theta_{\|}}{(m_2^2)} - \frac{\sin^2 \Theta_{\|}}{m_2^2}\right] \sin^2 \Theta_{\|}
\]

\[
(m_1^2 - m_2^2) = \frac{m_1^4 - m_2^4}{m_2^2} \sin^2 \Theta_{\|}
\]

\[
= \left(m_1^2 + m_2^2\right) \left(m_1^2 - m_2^2\right) \sin^2 \Theta_{\|}
\]

\[
\frac{m_2}{m_1^2 + m_2^2} = \sin \theta_{\|}
\]

\[
\Rightarrow \sin \Theta_{\|} = \frac{1}{\sqrt{1 + \frac{m_1^2}{m_2^2}}}
\]

\[
\Rightarrow \tan \theta_{\|} = \frac{m_2}{m_1}
\]