Asymptotic Stability of a Rate Control System with Communication Delays

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Abstract

We study the issue of asymptotic stability of a family of rate allocation algorithms with communication delays between network elements, and extend earlier results: First, we provide a lower bound on the convergence rate with a family of well known utility and resource price functions when *delay independent* stability conditions hold. Second, we derive *delay dependent* stability conditions with the same utility and resource price functions when a finite upper bound is known on the feedback delay. Numerical examples are provided to validate our analyses.

I. Introduction

Recently there has been much interest in understanding the stability property of a family of rate control schemes, called primal algorithms, in the presence of communication delays [3], [6], [9], [10], [15]. The primal algorithms, first proposed by Kelly *et al.* [8], are motivated by an optimization framework for a rate allocation with elastic traffic sources where the objective of the system is to maximize the aggregate utility of the users. Tan and Johari [6] studied the case where flows have the same round-trip delays and the same log utility functions, and provided local stability conditions in term of users' gain parameters and communication delays. Their results state that the system is locally stable if the product of gain parameter and communication delay is no larger than some specified constant. Similar results have been obtained for single flow and single resource cases with more general utility functions in [3], [9] and for single bottleneck with multiple heterogeneous users cases in [1].

In another set of work Ranjan *et al.* [10] and Ying *et al.* [15] studied the stability of the rate control system in the presence of *arbitrary* fixed delays between network elements, *e.g.*, network resources and end users, and *arbitrary* gain parameters of the end users. These approaches are consistent with the philosophy that, given the complexity and scale of the Internet, network protocols must be *simple* and *robust*. They derived sufficient conditions on users' utility and resource price functions for asymptotic (global) stability of the rate control system with arbitrary fixed communication delays and users' gain parameters. We refer to these conditions as *delay independent* stability conditions.

In this paper we study the same class of rate control systems investigated in [10], [15] and extend their results in two directions:

- We study *discretized* systems that approximate the delay differential systems studied in [10], [15]. Using these discretized systems we derive a lower bound on the convergence rate of the rate control system with a family of utility and resource price functions studied in [10], [15] under the delay independent stability conditions (Section V). The role of the maximum delay in feedback loop is highlighted. A closed form expression of a lower bound on the local convergence rate is also obtained.
- We study a simple network with a single flow traversing a single resource and investigate its stability when a finite upper bound on a feedback delay is known in advance. We derive a stability condition that hints at how an increasing feedback delay affects the stability condition. We refer to this condition as a *delay dependent* stability condition to distinguish it from the delay independent stability conditions. The derived condition is consistent with the earlier stability conditions in two extreme cases; when there is no feedback delay, the system is always stable with the employed utility and resource price functions. When the delay is allowed to be arbitrarily large, we recover the delay independent stability condition in [10], [15].

This paper is organized as follows: Section II describes the optimization framework for rate control. Section III describes the rate control system model with fixed communication delays. We outline the basic approach to the problem and summarize our previous work in Section IV. A lower bound on the convergence rate with the family of utility and price functions studied in [10], [15] is derived in Section V. Delay dependent stability conditions with the same utility and price functions employed in [10], [15] are obtained in Section VI. Numerical examples are presented in Section VII. We conclude in Section VIII.

II. BACKGROUND

In this section we briefly describe the rate control problem in the proposed optimization framework. Consider a network with a set $\mathcal{L}$ of resources or links and a set $\mathcal{I}$ of users. Let C_l denote the finite capacity of link $l \in \mathcal{L}$. Each user has a fixed route r_i, which is a non-empty subset of $\mathcal{L}$. We define a zero-one matrix A, where $A_{i,l} = 1$ if link l is in user i's route r_i and $A_{i,l} = 0$ otherwise. When the throughput of user i is x_i, user i receives utility $U_i(x_i)$. This utility function could represent either the user's true utility or some function assigned to the user through the selected end user algorithm. We take the latter view and assume that the utility functions of the users are used to select the desired rate allocation among the users. The utility $U_i(x_i)$ is an increasing, strictly concave and continuously differentiable function of x_i over the range $x_i \geq 0$. The rate control problem can be formulated as the following optimization problem [7]:

SYSTEM(U,A,C):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in \mathcal{I}} U_i(x_i) \\ \text{subject to} \quad & A^T x \leq C \ , \ x \geq 0 \end{aligned} \tag{1}$$

where $C = (C_l, l \in \mathcal{L})$.[1] The first constraint is the capacity constraint.

Assume that every user adopts rate-based flow control. Let $w_i(t)$ and $x_i(t)$ denote user i's willingness to pay per unit time and rate at time t, respectively.[2] Now suppose that at time t

[1] All vectors are assumed to be column vectors.

[2] Throughout the rest of the paper we refer to the willingness to pay per unit time as simply willingness to pay.

each resource $l \in \mathcal{L}$ charges a price per unit flow of $\mu_l(t) = p_l(\sum_{i:l\in r_i} x_i(t))$, where $p_l(\cdot)$ is an increasing function of the total rate going through it. Consider the system of differential equations

$$\frac{d}{dt}x_i(t) = \kappa_i \left(w_i(t) - x_i(t) \sum_{l \in r_i} \mu_l(t) \right) \ , \tag{2}$$

where $w_i(t) = x_i(t) \cdot U_i^{'}(x_i(t))$. For further explanation of (2), see [8]. Since we assume that the utility functions of the users are selected to decide the rate allocation among the users, under (2) one can see that both users' utility functions and resource price functions can be utilized to decide the operating point of the system. Therefore, the design of rate control algorithms is equivalent to selecting users' utility functions and the price functions of the resources in the network.

Kelly *et al.* [8] have shown that under some conditions on $p_l(\cdot), l \in \mathcal{L}$, the above system of differential equations converges to a point that maximizes the following expression

$$\mathcal{U}(x) = \sum_i U_i(x_i) - \sum_l \int_0^{\sum_{i:l\in r_i} x_i} p_l(y)dy. \tag{3}$$

Note that the first term in (3) is the objective function in our *SYSTEM(U, A, C)* problem. Thus, the algorithm proposed by Kelly *et al.* solves a relaxation of the *SYSTEM(U, A, C)* problem.

III. Network model with delays

In this section we first describe the network model that was used in [10] to capture the delays between the network resources and end users under the assumption that the delays are constant. Although in practice the delays are time varying due to time varying queue sizes, we assume that the variation in the delays due to fluctuating queue sizes is not significant. For example, AQM mechanisms that attempt to either maintain very small queue sizes, *e.g.*, AVQ, or keep the queue sizes around some target queue sizes, *e.g.*, REM, can be well approximated by our model. The definitions of the variables introduced throughout the paper are provided in Table I in the appendix.

We consider a set $\mathcal{I} = \{1, \ldots, N\}$ of users that share a network with a set $\mathcal{L} = \{1, \ldots, L\}$ of resources as described in Section II. Define $I_l := \{i \in \mathcal{I} \mid l \in r_i\}$, i.e., the set of users

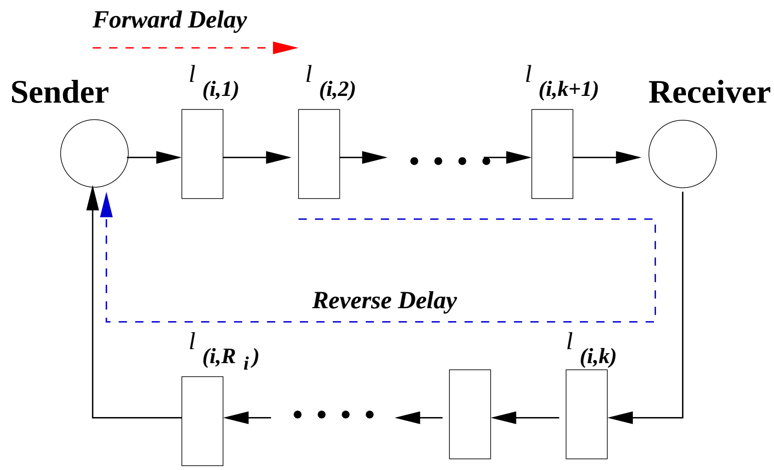

Fig. 1. Network model with delays.

traversing resource $l \in \mathcal{L}$. We assume that the price functions $p_l(\cdot), l \in \mathcal{L}$, are strictly increasing and continuous.

Both user rates from the senders to the resources and the feedback information from the resources to the users, which is typically carried by acknowledgments (ACKs), are delayed due to link propagation and transmission delays. For all $i \in \mathcal{I}$ and $l \in r_i$ let $T^f_{i,l}$ denote the forward delay user i's packets experience before reaching resource l from the sender and $T^r_{i,l}$ the reverse delay of the feedback signal from resource l to user i. If $l \notin r_i$, we assume that $T^r_{i,l} = T^f_{i,l} = 0$. Suppose that the links in $r_i = \{l_{i,1}, \ldots, l_{i,R_i}\}$ are arranged in the order user i's packets visit, where $R_i = |r_i|$ denotes the cardinality of r_i. Define T_i to be user i's round-trip delay, i.e., the sum of forward and reverse delays $T^f_{i,l_{i,k}} + T^r_{i,l_{i,k}}$, $k = 1, \ldots, R_i$. Under this general model, the end user dynamics are given by

$$\frac{d}{dt} x_i(t) = \kappa_i \left(x_i(t) U_i'(x_i(t)) - x_i(t - T_i) \Big(\sum_{l \in r_i} \mu_l(t - T^r_{i,l}) \Big) \right) \tag{4}$$

where $\mu_l(t - T^r_{i,l}) = p_l\Big(\sum_{j \in I_l} x_j(t - (T^r_{i,l} + T^f_{j,l})) \Big)$. Note that the price of resource l at time t depends on the rates of the users at time $t - T^f_{j,l}$ due to the delay from the senders to the resource. The feedback signal generated by the resource is then delayed by $T^r_{i,l}$ before sender i receives it.

IV. An outline of approach & previous work

This section first describes the basic set-up that will be used throughout the paper and then summarizes our earlier work reported in [10].

A. Set-up and assumptions

Since the rate of a user is limited in practice, due to for example the receiver buffer size, we assume that a user's rate belongs to a compact set.

Assumption 1: A user rate belongs to a compact set $\mathcal{X} := [X_{min}, X_{max}] \subset \mathbb{R}_+ := (0, \infty)$.

The upper bound X_{max} can be arbitrarily large, while the lower bound X_{min} can be arbitrarily close to zero.

Given the delays $T^r_{i,l}$ and $T^f_{i,l}$, feedback information from the resources arriving at a sender cannot be delayed by more than some finite constant. We define this upper bound on the information delay to be

$$T_{max} := \max_{i \in \mathcal{I}, l \in r_i} (\max_{j \in I_l} T^r_{i,l} + T^f_{j,l}) \tag{5}$$

One can verify that the delay differential equations in (4) do not depend on the system state before $t - T_{max}$. In other words, T_{max} plays the role of an upper bound r on the feedback delay in the framework of Hale and Lunel in [4, pp. 38-48].

We first define for each user $i \in \mathcal{I}$ a function that gives us its willingness to pay as a function of its current rate:

$$y_i = x_i \cdot U'_i(x_i) =: g_i(x_i) \tag{6}$$

In order to facilitate our analysis, following the same steps in [10], we rewrite (4) in terms of users' willingness to pay $y_i(t) = g_i(x_i(t))$. The notation needed to capture the complex dependency of the delay differential equations on the past willingness to pay of the users becomes cumbersome. However, the final form (given by (11)) simply states that users continually update their willingness to pay based on the *finite* history of their willingness to pay.[3]

$$\frac{d}{dt}y_i(t) = \kappa_i g_i^{'}(g_i^{-1}(y_i(t)))\Big[y_i(t) - g_i^{-1}(y_i(t-T_i))\sum_{l\in r_i} p_l\Big(\sum_{j\in I_l} g_j^{-1}(y_j(t-(T^r_{i,l}+T^f_{j,l})))\Big)\Big]$$
$$= \kappa_i g_i^{'}(g_i^{-1}(y_i(t)))\left[y_i(t) - \check{f}_i\Big(\overline{g}^{-1}(\overline{y}(t-T_i)), (\overline{g}^{-1}(\tilde{y}_{(i,l)}(t)), l\in r_i)\Big)\right] \quad (7)$$

where $\overline{y}(t) = (y_i(t), i\in\mathcal{I})$,[4] $\overline{g}^{-1}(\overline{y}) = (g_i^{-1}(y_i), i\in\mathcal{I})$ is the component-wise inverse,

$$\tilde{y}_{(i,l)}(t) = \Big(y_j(t-T^r_{i,l}-T^f_{j,l}), j\in\mathcal{I}\Big)\ ,\ l\in r_i \quad (8)$$

and

$$\check{f}_i\left(\overline{g}^{-1}(\overline{y}(t-T_i)), (\overline{g}^{-1}(\tilde{y}_{(i,l)}(t)), l\in r_i)\right)$$
$$= \check{f}_i\left(\overline{g}^{-1}(\overline{y}(t-T_i)), \overline{g}^{-1}(\tilde{y}_{(i,l_{i,1})}(t)), \ldots, \overline{g}^{-1}(\tilde{y}_{(i,l_{i,R_i})}(t))\right)$$
$$:= g_i^{-1}(y_i(t-T_i))\left[\sum_{l\in r_i} p_l\Big(\sum_{j\in I_l} g_j^{-1}(y_j(t-T^r_{i,l}-T^f_{j,l}))\Big)\right]\ . \quad (9)$$

Note that (9) gives user i's total price at time t as a function of the past willingness to pay of the users (and hence their rates through the relationship (6)).

Let us define

$$Y^i(t) = \Big(\overline{y}(t-T_i), \tilde{y}_{(i,l_{i,1})}(t), \ldots, \tilde{y}_{(i,l_{i,R_i})}(t)\Big)\ \in \mathbb{R}^{N\cdot(R_i+1)}\ ,$$
$$Y(t) = \Big(Y^1(t), \ldots, Y^N(t)\Big)\ \in \mathbb{R}^{N\cdot\sum_{i\in\mathcal{I}}(R_i+1)}\ , \quad (10)$$

and $G_i(Y^i(t)) = \Big(\overline{g}^{-1}(\overline{y}(t-T_i)), \overline{g}^{-1}(\tilde{y}_{(i,l_{i,1})}(t)), \ldots, \overline{g}^{-1}(\tilde{y}_{(i,l_{i,R_i})}(t))\Big)$. We write (7) in a more compact matrix form:

$$\frac{d}{dt}\overline{y}(t) = \mathbf{K}(\overline{y}(t))\left[\check{\mathbf{F}}(Y(t)) - \overline{y}(t)\right] \quad (11)$$

where

$$\check{\mathbf{F}}_i(Y(t)) := \check{f}_i(G_i(Y^i(t)))\ , \quad (12)$$

and $\mathbf{K}(\cdot)$ is the state dependent diagonal gain matrix with

$$\mathbf{K}_{ii}(\overline{y}(t)) = -\kappa_i g_i^{'}(g_i^{-1}(y_i(t)))\ . \quad (13)$$

[3]For now we assume that (i) the rates do not hit the boundaries X_{min} and X_{max} and (ii) the inverses $g_i^{-1}(\cdot)$ exist. We will come back to these assumptions shortly.

[4]We will use an overline to denote a vector in the rest of the paper.

As mentioned earlier, it is clear from (11) that users continually attempt to reach an equilibrium where their prices $\check{\mathbf{F}}_i(Y(t))$ equal their willingness to pay $y_i(t)$.

We make the following assumption on $g_i(\cdot), i \in \mathcal{I}$, and the resource price functions $p_l(\cdot), l \in \mathcal{L}$.

Assumption 2: (i) The function $g_i(x_i)$ is strictly decreasing with $g_i^{'}(x_i) < 0$ for all $x_i > 0$, (ii) the price functions $p_l(x)$ are strictly increasing in x for all $l \in \mathcal{L}$, and (iii) $g_i(x_i)$ is Lipschitz continuous on $[X_{min}, X_{max}]$ and $p_l(x)$ is Lipschitz continuous on $[|I_l| \, X_{min}, |I_l| \cdot X_{max}]$.

This assumption ensures that the state dependent gain matrix $\mathbf{K}(\cdot)$ is positive definite. Assumption 2(i) implies that the marginal utility decreases faster than x_i^{-1} from the definition of $g_i(x_i)$. Hence, a change in price per unit flow does not cause a large change in user demand and user demands are inelastic. It also guarantees the existence of $g_i^{-1}, i \in \mathcal{I}$. Therefore, the convergence of $\overline{y}(t)$ to $\overline{y}$ implies the convergence of $\overline{x}(t) = \overline{g}^{-1}(\overline{y}(t))$ to $\overline{x} = \overline{g}^{-1}(\overline{y})$.

B. *Previous work*

We have investigated the stability issue of the system in (4) with fixed delays with no queueing delays in [10]: Define $\Xi := \sum_{i \in \mathcal{I}}(R_i + 1)$. Suppose that there exists a sequence of closed, convex product spaces $D_k = \prod_{i=1}^{N} \text{proj}_i(D_k) \subseteq \mathbb{R}_+^N, k \geq 0$, where $\text{proj}_i(D_k)$ denotes the projection of the i-th component of D_k, such that $\check{\mathbf{F}}(D_k^{\Xi}) \subset int(D_{k+1}) \subset D_{k+1} \subset int(D_k)$ and $\cap_{k \geq 0} D_k = \{\overline{y}^{\star}\}$, where $int(D_k)$ denotes the interior of D_k and $\overline{y}^{\star}$ denotes the vector of users' willingness to pay at the equilibrium $\overline{x}^{\star}$. We have proved that under this assumption the delay differential system with fixed delays is asymptotically stable if the initial function lies in the Banach space of the continuous functions mapping the interval $[-T_{max}, 0]$ to D_0, where T_{max} is defined in (5). Moreover, we have shown that the delay differential system in (4) with arbitrary fixed delays, is stable *if* the system with a *homogeneous* delay, in which there is no forward delay from the senders to the resources and the reverse path delays are all given by the *same* constant T, is stable.

We applied this result to derive a sufficient condition for asymptotic global stability with well known isoelastic (or constant elasticity of substitution) utility functions and resource price functions [2]: The class of users' utility functions is assumed to be of the form

$$U_i(x) = -\frac{1}{a_i}\frac{1}{x^{a_i}} \ , \quad a_i > 0 \ . \tag{14}$$

A slight variation of this class of utility function is given by

$$U_i(x) = -\frac{x^{-a_i} - 1}{a_i} \ , \quad a_i > 0 \ . \tag{15}$$

With these utility functions the price elasticity of demand, which measures how responsive the demand is to a change in price and is defined to be the percent change in demand divided by the percent change in price [13], is given by

$$\frac{p}{x_i^\star(p)}\frac{dx_i^\star(p)}{dp} = \frac{p}{p^{-\frac{1}{1+a_i}}} \cdot \frac{-1}{1+a_i} p^{-\frac{1}{1+a_i}-1} = \frac{-1}{1+a_i} \ , \tag{16}$$

where $x_i^\star(p)$ is the unique optimal rate that maximizes the net utility $U_i(x) - x \cdot p$ given a price per unit flow p. Note that the price elasticity of demand in (16) does not depend on the operating point $x_i^\star(p)$, hence the name isoelastic utility functions. The price elasticity of demand decreases with a_i.[5] It is easy to see that the utility functions in (14) and (15) satisfy Assumption 2.

The class of resource price functions that we employ is of the form

$$p_l(y) = c_l \cdot \left(\frac{y}{C_l}\right)^{b_l} \ , \tag{17}$$

where $b_l > 0$, c_l is some positive constant, and C_l is the capacity of resource $l \in \mathcal{L}$. However, C_l can be replaced with any positive constant, *e.g.*, virtual capacity in AVQ. The parameter b_l is used to change the shape of the price function.

Suppose that each user i has a utility function of either (14) or (15) with parameter a_i and the resource price function $p_l(\mu)$ is given by (17) with parameter $b_l > 0$. Under this assumption the system in (4) is proved to be asymptotically globally stable if $a_i > 1 + \max_{l \in r_i} b_l$ for all $i \in \mathcal{I}$, starting with any continuous initial function bounded away from zero. Similar results are reported in another independent study [15].

In this paper we extend the results obtained in [10], [15] in two directions: First, we derive a lower bound on the convergence rate of the system with the utility and price functions of

[5] When comparing the price elasticity, typically the absolute value of (16) is used.

the form (14) (or (15)) and (17), respectively, under the assumption that the stability condition holds, i.e., $a_i > 1 + \max_{l \in r_i} b_l$ for all $i \in \mathcal{I}$ (Section V). Secondly, we consider the case where a finite upper bound on feedback delay is known and derive *delay dependent* asymptotic stability conditions for the utility and resource price functions of (14) and (17) in single-flow, single-resource cases (Section VI). The derived conditions reveal how the known finite upper bound on the delay affects the global stability. In two extreme cases - either no delay or an arbitrary delay - we recover the earlier known stability conditions.

V. A LOWER BOUND ON CONVERGENCE RATE

In this section we first introduce a family of discretized models and establish asymptotic stability conditions similar to those derived in [10]. These discretized models in fact capture the packet level dynamics more faithfully than the continuous time model in (4). Then, we adopt the same family of utility and resource price functions studied in [10], [15] (i.e., (14) and (17)) and derive a lower bound on the convergence rate of the delay differential system in (4) with fixed delays. This is done by considering a sequence of discretized models with decreasing step sizes. The delay differential system can be obtained as a limit as the step size approaches zero.

A. Discretized models

We assume that the time is divided into contiguous timeslots, where a timeslot can be interpreted to represent the smallest time scale at which network dynamics evolve. We denote the duration of a timeslot by $\delta > 0$, and the delays $T^r_{i,l}$ and $T^f_{i,l}$ for all $i \in \mathcal{I}$ and $l \in r_i$, are assumed to be integer multiples of δ.[6]

We first replace the time derivative $\frac{d}{dt}\overline{y}(t)$ in (7) with an approximation $(\overline{y}(t+\delta) - \overline{y}(t))/\delta$:

$$\begin{aligned} \frac{\overline{y}(t+\delta) - \overline{y}(t)}{\delta} &= \mathbf{K}(\overline{y}(t)) \left[g_i^{-1}(y_i(t - T_i)) \sum_{l \in r_i} p_l \Big(\sum_{j \in I_l} g_j^{-1}(y_j(t - T^r_{i,l} - T^f_{j,l})) \Big) - y_i(t) \right] \\ &= \mathbf{K}(\overline{y}(t)) \left[\check{f}_i(\overline{g}^{-1}(\overline{y}(t - T_i)), (\overline{g}^{-1}(\tilde{y}_{(i,l)}(t)), l \in r_i)) - y_i(t) \right] \end{aligned} \quad (18)$$

[6] When this assumption does not hold, one can apply a similar argument used in the following subsection to obtain a lower bound on convergence rate directly to the continuous time differential system given by (7).

where $\tilde{y}_{(i,l)}(t)$ is defined in (8), and $\check{f}_i(\cdot)$ is defined in (9).

Eq. (18) can be rewritten as

$$\overline{y}(t+\delta) = (I - \delta \cdot \mathbf{K}(\overline{y}(t))) \cdot \overline{y}(t) + \delta \cdot \mathbf{K}(\overline{y}(t)) \cdot \check{\mathbf{F}}(Y(t)) \ , \tag{19}$$

where I is the $N \times N$ identity matrix, and $Y(t)$ is defined in (10). Note that we discretize (7) instead of (4).

We assume that the interarrival times of the feedback signal are δ and the acknowledgments from the receivers carrying feedback information arrive at the senders at discrete times $t_n = n \cdot \delta, n \geq 0$. The updates of transmission rates take place upon the arrivals of acknowledgments. Each acknowledgment is assumed to contain the precise value of the resource prices. The effects of finite granularity of feedback information are discussed in [11], [12].

Define $\alpha := \delta^{-1}$. We rewrite the continuous time system in (19) as the following discretized system with unit time δ:

$$\overline{y}_{n+1} = (I - \delta \cdot \mathbf{K}(\overline{y}_n)) \cdot \overline{y}_n + \delta \cdot \mathbf{K}(\overline{y}_n) \cdot \check{\mathbf{F}}(\tilde{Y}_n) \ , \tag{20}$$

where

$$\begin{aligned} \tilde{y}^n_{(i,l)} &= \left(y_{n-\alpha(T^r_{i,l}+T^f_{j,l}),j}, \ j \in \mathcal{I} \right) \ , \ l \in r_i \\ \tilde{Y}^i_n &= \left(\overline{y}_{n-\alpha \cdot T_i}, \tilde{y}^n_{(i,l_{i,1})}, \ldots, \tilde{y}^n_{(i,l_{i,R_i})} \right) \ , \text{ and } \quad \tilde{Y}_n = \left(\tilde{Y}^1_n, \ldots, \tilde{Y}^N_n \right) \end{aligned} \tag{21}$$

We define the invariance and a fixed point of the map $\check{\mathbf{F}}(\cdot)$ as follows.

Definition 1: A set $D \subset \mathbb{R}^N_+$ is said to be invariant under the map $\check{\mathbf{F}}(\cdot)$ defined in (12) if $\check{\mathbf{F}}(Y) \in D$ whenever $Y \in D^\Xi$, i.e., $Y = (Y^1, \ldots, Y^N)$ and $Y^i \in D^{R_i+1}$ for all $i \in \mathcal{I}$. A vector $\overline{y}^\star \in \mathbb{R}^N_+$ is said to be a fixed point of $\check{\mathbf{F}}(\cdot)$ if $\check{\mathbf{F}}(\overline{y}^\star, \ldots, \overline{y}^\star) = \overline{y}^\star$.

The invariance of the map $\check{\mathbf{F}}(\cdot)$ can be interpreted as follows. Suppose that users' willingness to pay $\overline{y}(t)$ at time t as well as time delayed values $\hat{y}_{(i,l)}(t)$, $i \in \mathcal{L}$ and $l \in r_i$, belong to the set D. Then, the invariance of the set D implies that users' prices $\check{\mathbf{F}}(Y(t))$ stay within the set D. Similarly, a fixed point $\overline{y}^\star$ means that if $\overline{y}(t) = \overline{y}^\star$, $\hat{y}_{(i,l)}(t) = \overline{y}^\star$ for all $i \in \mathcal{I}$ and $l \in r_i$, then $\check{\mathbf{F}}(\overline{y}^\star, \ldots, \overline{y}^\star) = \overline{y}^\star$. In other words, $\overline{y}(t)$ remains constant.

One can verify that if $\overline{y}^\star$ is a fixed point of $\check{\mathbf{F}}(\cdot)$, then $\overline{g}^{-1}(\overline{y}^\star) = (g_1^{-1}(y_1^\star), g_2^{-1}(y_2^\star), \ldots, g_N^{-1}(y_N^\star))$ is a solution to (3) from (7) and (9), i.e., $\overline{y}^\star = \overline{g}(\overline{x}^\star)$ where $\overline{x}^\star$ is a solution to (3).

Assumption 3: Suppose that $D \subset \mathbb{R}_+^N$ is a closed, convex product space invariant under $\check{\mathbf{F}}(\cdot)$ in (12). In addition, assume that

$$\delta \cdot \mathbf{K}_{ii}(\overline{y}) < 1 \tag{22}$$

for all $i \in \mathcal{I}$ and for all $\overline{y} \in D$.

Assumption 3 is needed to ensure that the state-dependent gain $\gamma(\overline{y}, \delta) > 0$, where

$$\gamma_i(\overline{y}, \delta) = 1 - \delta \cdot \mathbf{K}_{ii}(\overline{y}) \ , \ i \in \mathcal{I} \ , \tag{23}$$

for all $\overline{y} \in D$. Although the form of condition (22) is similar to the stability conditions obtained in [6], [9], the reason for its introduction is quite different.

Since the delay differential system in (4) can be recovered as the limit of (18) as δ goes to zero, the assumption in (22) can be removed in the case of delay differential system. Given $\{\overline{y}_0, \overline{y}_{-1}, \ldots, \overline{y}_{-\alpha \cdot T_{max}}\}$ the solution $\overline{y}_n, \ n > 0$, of the discretized system can be computed by successively iterating the map given in (12) according to (20).

We show that, similar to the delay differential system studied in [10], the asymptotic stability of the discretized model in (20) can be described by the corresponding properties of the map $\check{\mathbf{F}}$ defined in (12).

Theorem 1: (Asymptotic stability) Assume (i) there is a sequence of closed, convex product spaces $D_k, k \geq 0$, such that $\check{\mathbf{F}}(D_k^{\Xi}) \subset int(D_{k+1}) \subset D_{k+1} \subset int(D_k)$ and $\cap_{k \geq 0} D_k = \{\overline{y}^\star\}$, where $\overline{y}^\star$ is a stable fixed point of the map $\check{\mathbf{F}}(\cdot)$, and (ii) $\gamma(\overline{y}, \delta) > 0$ for all $\overline{y} \in D_0$. Then, if $\overline{y}_n \in D_0$, $n = 0, -1, \ldots, -\alpha \cdot T_{max}$, we have $\overline{y}_n \in D_0$ for all $n \geq 0$ and $\lim_{n \to \infty} \overline{y}_n = \overline{y}^\star$.

Proof: The proof is the theorem is similar to that of Theorem 4 in [10] and is omitted. ∎

One thing to note in Theorem 1 is that the condition (i) is the same sufficient condition assumed for the asymptotic stability of the delay differential system in subsection IV-B. In fact, the same sequence of closed product spaces can be used for both cases.

B. Lower bound on convergence rate

In this subsection we study the discretized systems introduced in subsection V-A with the utility and resource price functions given by (14) and (17), respectively; the price functions of the resources are given by $p_l(x) = \left(\frac{x}{C_l}\right)^{b_l}, b_l > 0$, and the utility functions of the users are of $U_i(x) = \frac{-1}{a_i \cdot x^{a_i}}, a_i > 0$. We assume that $a_i > b^i_{max} + 1$ for all $i \in \mathcal{I}$, where $b^i_{max} = \max_{l \in r_i} b_l$. Also, in order to ensure that the equilibrium is an interior point, we assume that $C_l > |I_l|$.

Fix some $\overline{\alpha} > 1$, and let $\overline{\beta}$ be a constant that satisfies

$$\hat{\mathbf{F}}(\overline{\beta} \cdot \overline{x}^\star) < \overline{\alpha} \cdot \overline{x}^\star \text{ and } \overline{\beta} \cdot \overline{x}^\star < \hat{\mathbf{F}}(\overline{\alpha} \cdot \overline{x}^\star) \ , \tag{24}$$

where $\hat{\mathbf{F}}(\overline{x}) := \overline{g}^{-1}(f_i(\overline{x}))$ and $f_i(\overline{x}) := x_i \sum_{l \in r_i} p_l(\sum_{j \in I_l} x_j)$. It is shown [10, Lemma 1] that any $\overline{\beta}$ such that

$$\overline{\alpha}^{1/\sigma} < \overline{\beta} < \overline{\alpha}^{\sigma} \tag{25}$$

satisfies (24). Under the assumption $a_i > b^i_{max} + 1$ and $C_l > |I_l|$, the existence of $\overline{\beta}$ satisfying (24) is guaranteed by Lemma 1 in [10]. Define

$$\tilde{D}_k = \begin{cases} \prod_{i=1}^N [\overline{\alpha}^{\sigma^k} x^\star_i \ , \ \overline{\beta}^{\sigma^k} x^\star_i] \ , & k \text{ odd} \\ \prod_{i=1}^N [\overline{\beta}^{\sigma^k} x^\star_i \ , \ \overline{\alpha}^{\sigma^k} x^\star_i] \ , & k \text{ even} \end{cases} \ , \tag{26}$$

where σ is some negative constant satisfying $\max_{i \in \mathcal{I}} \frac{b^i_{max}+1}{a_i} =: \Delta < -\sigma < 1$ and $\prod$ denotes the Cartesian product space. Lemma 2 of [10] tells us that $\hat{\mathbf{F}}(\tilde{D}_k) \subset int(\tilde{D}_{k+1}) \subset \tilde{D}_{k+1} \subset int(\tilde{D}_k)$ and $\cap_{k=0}^{\infty} \tilde{D}_k = \{\overline{x}^\star\}$ under the assumed condition $\Delta < 1$.

Theorem 2: Assume (i) $a_i > b^i_{max} + 1$ for all $i \in \mathcal{I}$, and (ii) $\delta \cdot \mathbf{K}_{ii}(\overline{g}(\overline{x})) < 1$ for all $\overline{x} \in \tilde{D}_0$. If $\overline{x}_n \in \tilde{D}_0$, $n = 0, \ldots, -\alpha \cdot T_{max}$, then there exist $K > 0$ and $\psi(\delta) > 0$ such that,

$$\frac{|x_{n,i} - x^\star_i|}{x^\star_i} \leq K \cdot \exp(-\psi(\delta) \cdot n) \quad \text{ for all } i \in \mathcal{I} \text{ and for all } n = 0, 1, \ldots \tag{27}$$

Proof: A proof is provided in Appendix I. ∎

In the proof of the theorem we show that

$$K = \overline{K}/\sigma^2 \quad \text{ and } \quad \psi(\delta) = \frac{\phi}{M(\delta, \sigma) + \alpha \cdot T_{max}} \ ,$$

where

$$\overline{K} = \exp(\overline{\alpha}^{-1/\sigma} - 1) - 1 \quad \text{and} \quad \phi = \log(-1/\sigma) \ ,$$

satisfy the condition in (27) with any $\sigma \in I^\star := (-1, \ -\Delta)$. The constant $M(\delta, \sigma)$ is inversely proportional to δ, i.e., $M(\delta, \sigma) = M^\star/\delta = \alpha \cdot M^\star$ for some positive constant $M^\star := M^\star(\tilde{D}_0, \sigma)$ that increases with the *initial* invariant set $\tilde{D}_0$. Hence, $\psi(\delta)$ can be rewritten as $\frac{\phi \cdot \delta}{M^\star + T_{max}}$.

Fix $t \geq 0$ and let $n(t, \delta) = \lfloor t/\delta \rfloor$. For sufficiently small δ, we have $\psi(\delta) \cdot n(t, \delta) \cong \frac{\phi \cdot t}{M^\star + T_{max}}$. Since the continuous time model with fixed propagation delays in (7) is obtained as a limit of (20) as $\delta \downarrow 0$, one can see that $\lim_{\delta \downarrow 0} \psi(\delta) \cdot n(t, \delta) = \frac{\phi \cdot t}{M^\star + T_{max}}$ and

$$\sup_{\sigma \in I^\star} \frac{\phi}{M^\star(\tilde{D}_0, \sigma) + T_{max}} = \sup_{\sigma \in I^\star} \frac{\log(-1/\sigma)}{M^\star(\tilde{D}_0, \sigma) + T_{max}} \tag{28}$$

provides a *lower* bound on the convergence rate of the delay differential system in [10], which clearly depends on the initial invariant set $\tilde{D}_0$ through $M^\star(\tilde{D}_0, \sigma)$. Furthermore, as the initial invariant set $\tilde{D}_0$ becomes smaller (i.e., $\tilde{D}_0 \downarrow \{\overline{x}^\star\}$), the constant $M^\star(\tilde{D}_0, \sigma)$ converges to

$$\underline{M} := \min_{i \in \mathcal{I}} \frac{a_i(1 - \Delta^2)}{\mathbf{K}_{ii}(\overline{y}^\star) \cdot \Delta} \ . \tag{29}$$

Note that (29) does not depend on the selected value of $\sigma \in I^\star$. Hence, under the stability conditions in Theorem 2, as $\overline{x}(t)$ gets close to the solution $\overline{x}^\star$, the convergence rate approaches or becomes larger than

$$\frac{\log(\Delta^{-1})}{\underline{M} + T_{max}} \ . \tag{30}$$

This clearly highlights the dependence of the lower bound on T_{max}.

Interestingly, the lower bound is monotonically increasing in users' gain parameters κ_i, $i \in \mathcal{I}$, as $\underline{M}$ is decreasing in users' gain parameters and neither $\overline{x}^\star$ nor Δ depends on them. It is clear

$$\lim_{\kappa_i \uparrow \infty} \frac{\log(\Delta^{-1})}{\underline{M} + T_{max}} = \frac{\log(\Delta^{-1})}{T_{max}} \ . \tag{31}$$

In fact, for any fixed $\tilde{D}_0$, $M^\star(\tilde{D}_0, \sigma)$ goes to 0 as $\kappa_i, i \in \mathcal{I}$, go to ∞ and the limit of the lower bound in (28) is also given by $\log(\Delta^{-1})/T_{max}$.

Numerical examples of the bounds in (27) with the solutions of (4) are provided in Section VII. They show that the accuracy of the lower bound generally improves with users' gain parameters.

Moreover, for all sufficiently large users' gain parameters, the behavior of the solutions is similar, and the bounds in (31) provide close bounds.

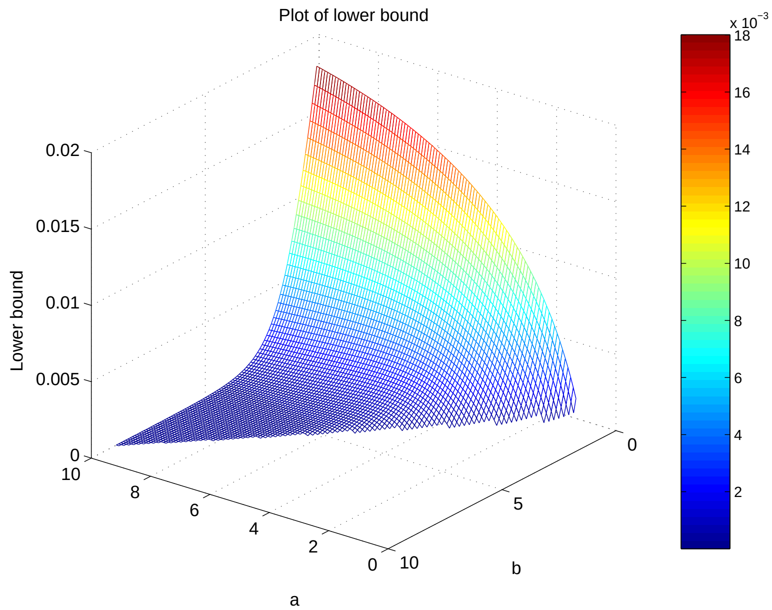

Fig. 2. Plot of lower bound $\log(\Delta^{-1})/(\underline{M}+T_{max})$ as a function of a and b.

We plot the lower bound $\log(\Delta^{-1})/(\underline{M}+T_{max})$ around the equilibrium in a simple case of a single flow traversing a single resource in Fig. 2. The gain parameter κ is set to 0.5, the round-trip delay of the flow is 100, and the capacity $C = 10$. We vary the utility parameter a and the price function parameter b to see how these parameters affect the lower bound on the convergence rate. As expected, for a fixed value of b the lower bound increases with a, whereas it decreases with b for a fixed value of a. One thing to notice is that the increase in a is concave, while the decrease in b is convex.

VI. Delay dependent stability condition: a single flow, a single resource case

Our stability results presented in the previous sections are concerned with the case where *arbitrary* delays $T^r_{i,l}$ and $T^f_{i,l}$ are allowed. However, if a *finite* upper bound on the delays is known in advance, less stringent stability conditions may suffice to ensure stability. In this section we consider the case of a single flow utilizing a single resource with feedback delay present only in the reverse path. Although we only consider the utility and price functions of the form (14) and (17), the basic idea used here can be applied to more general functions. We derive a delay *dependent* stability condition that highlights the relation between the feedback delay, gain parameter, and the stability of the system.

Denote the utility parameter of the user and the price parameter of the resource by a and b, respectively. The feedback delay is given by $T > 0$. Assume that the initial function ϕ_x belongs to $\mathcal{C}([-T, 0], \mathbb{R}_+)$. We are interested in the case where the delay independent stability condition does not hold, whence we assume $a < 1 + b$.

Theorem 3: Suppose that

$$\frac{1+b}{a} \left(1 - \exp(-\kappa \cdot \Theta(T, \kappa))\right) < 1 \ , \tag{32}$$

where $\Theta(T, \kappa) = a \cdot T \cdot v(T)^{1+\frac{1}{a}}$, and $v(t), t \geq 0$, is the solution to the initial value problem of

(i) $v(0) = y^\star = C^{-\frac{ab}{1+a+b}}$, and

(ii) $\frac{d}{dt} v(t) = \mathbf{K}(v(t))(\mathbf{F}(y_*) - v(t))$ with

$$y_* = \left(C^{\frac{b(1+a)}{1+a+b}} + (1+a)\kappa \cdot T \right)^{-a/(1+a)} \quad .$$

Then, $x(t; \phi_x)$ generated by (4) with an initial function $\phi_x \in \mathcal{C}([-T, 0], \mathbb{R}_+)$ satisfies $x(t; \phi_x) \to x^\star$ as $t \to \infty$, i.e., the system is globally asymptotically stable.

Proof: The proof is provided in Appendix II. ∎

Note that, for any fixed value of $\kappa > 0$, $\lim_{T \downarrow 0} \Theta(T, \kappa) = 0$ and $\lim_{T \uparrow \infty} \Theta(T, \kappa) = \infty$. Therefore, one can see that the condition (32) holds for *all* values of $a > 0$ and $b > 0$ in the absence of delay T because $\frac{1+b}{a}(1 - \exp(-\kappa \cdot \Theta(0, \kappa))) = \frac{1+b}{a} \cdot (1 - 1) = 0 < 1$. Similarly, the condition holds for all values of $T \geq 0$ only if $\frac{1+b}{a} < 1$, whence recovering our *delay independent* stability result in [10]. Therefore, the condition (32) in Theorem 3 hints at how the upper bound on the feedback delay affects the global stability condition. We plot the maximum delay T that satisfies condition (32) as a function of the utility and price function parameters a and b in Fig. 3. Here we only plot the value for the cases where $a \leq 1 + b - 10^{-1}$. The capacity $C = 10$, and the gain $\kappa = 0.2$. It is clear from the figure that as a approaches $1 + b$, the lower bound on the maximum stable delay increases rapidly.

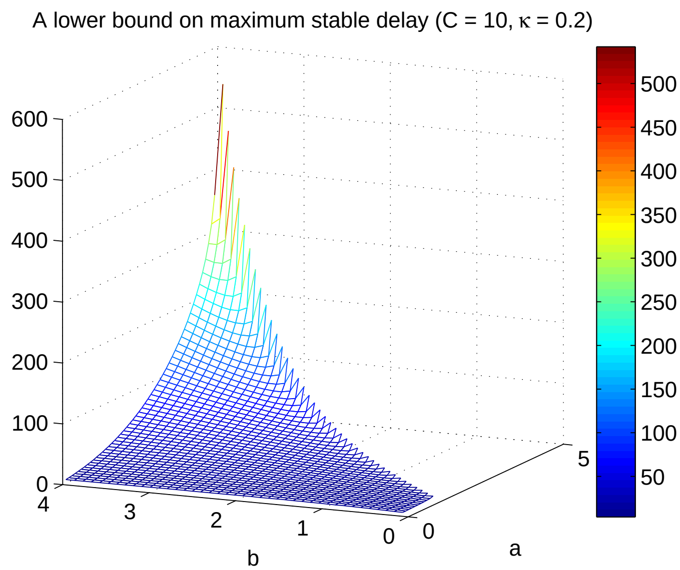

Fig. 3. Plot of the maximum stable delay according to condition (32).

Similar delay dependent conditions are also reported in [3], [9] in the same setting of a single flow and a single resource. The authors of [9] consider the case where $w(t) = w > 0$ for all $t \geq 0$. This represents the case with the utility function given by $w \cdot \log(x)$. They show that under some conditions the system is stable if $T \cdot \kappa \leq \frac{1}{4}$. The stability conditions for more general utility and resource price functions are derived in [3, Theorem II.1]. The stability conditions derived in [3] depend on the size of the domain $\mathcal{X}_0 = [x_{lb}, x^{ub}]$, in which the user rate $x(t)$ lies after some finite time t_0, i.e., $x(t) \in \mathcal{X}_0$ for all $t \geq t_0$. It states that if $\kappa \cdot T < \frac{A(a,b,\mathcal{X}_0)}{B(a,b,\mathcal{X}_0)}$, where $A(\cdot)$ and $B(\cdot)$ are some functions, then the system is globally exponentially stable. However, with the utility and price functions of (14) and (17), respectively, the stability condition in [3, p. 1057] yields lower bounds on the maximum stable feedback delays smaller than those shown in Fig. 3 over the same set of parameters.

VII. Numerical examples

In this section we present numerical examples with the solutions of (4) and the corresponding bounds in (30). There are two homogeneous flows sharing a single resource. The utility function parameter and the price function parameter are set to 4.1 and 3, respectively, so that they satisfy the delay independent stability condition $a = 4.1 > 4 = 1 + b$. Feedback delay exists only in the reverse path from the resource to the senders and is set to 500 for both flows. Initial function $\phi_x(s) = [3\ 3]^T$ for all $s \in$ [-500, 0]. The constant α is selected to be $3/x_1^\star = 3/1.404$, and $\beta = \alpha^{-1/\Delta} + 10^{-4}$ $(< \alpha^{-\Delta})$.

We vary the gain parameters $\kappa_i, i = 1, 2$, and plot $x_1(t)$ and the bounds calculated using (30) in Fig. 4. With $\kappa_i = 0.1, i = 1, 2$, we have $\underline{M} = 2.7875$, which is much smaller than $T_{max} = 500$. Therefore, the bounds given by (30) and (31) are approximately the same. Moreover, when T_{max} is large, the behavior of the solution is similar for all reasonably large gain parameters, with the solution becoming more square looking with increasing gain parameter. This is shown in Fig. 5. Furthermore, Fig. 5(b) shows that our bounds become tight as $t \to \infty$ for sufficiently large gain parameters.

VIII. Conclusions

In this paper we studied the problem of designing a robust congestion control mechanism in the presence of communication delays between end users and network resources. We first provided a lower bound on the convergence rate of a family of rate control schemes when the delay independent stability conditions hold. Secondly, we studied the delay-dependent stability and derived a sufficient condition for the asymptotic stability when there is a finite upper bound on a communication delay.

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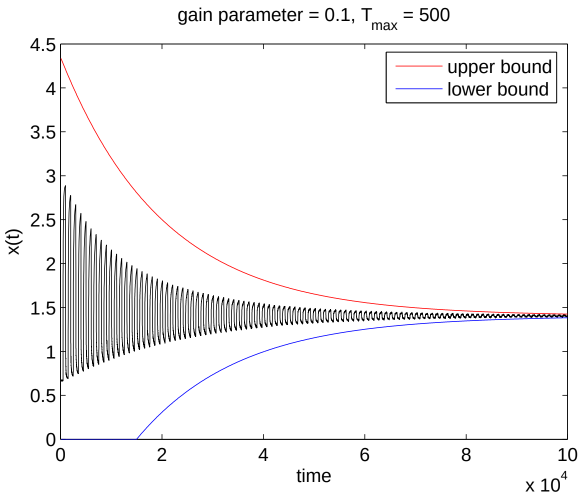
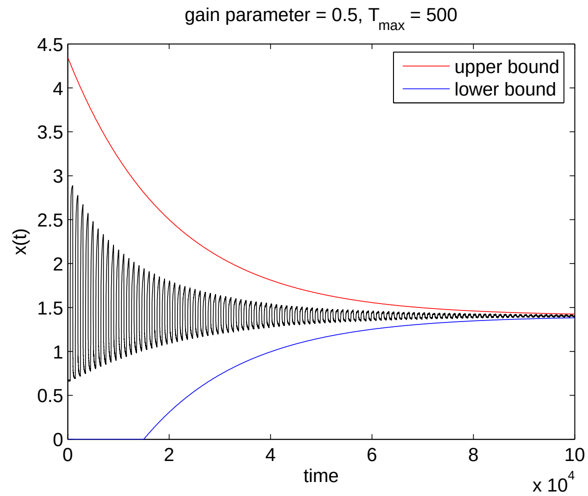
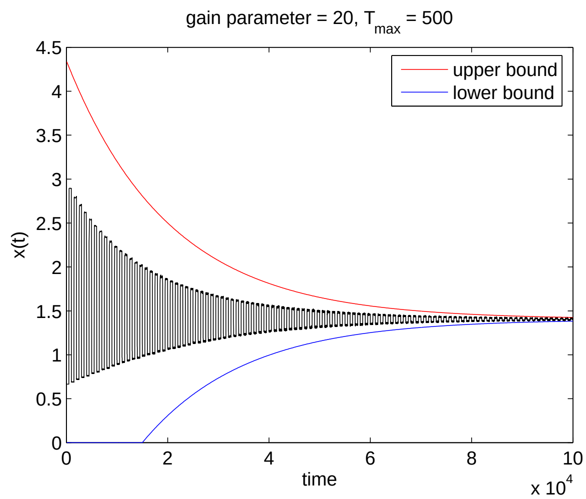


(a)



(b)



(c)

Fig. 4. The evolution of $x_1(t)$ and the bounds. (a) $\kappa_i = 0.1$, (b) $\kappa_i = 0.5$, and (c) $\kappa_i = 20$.

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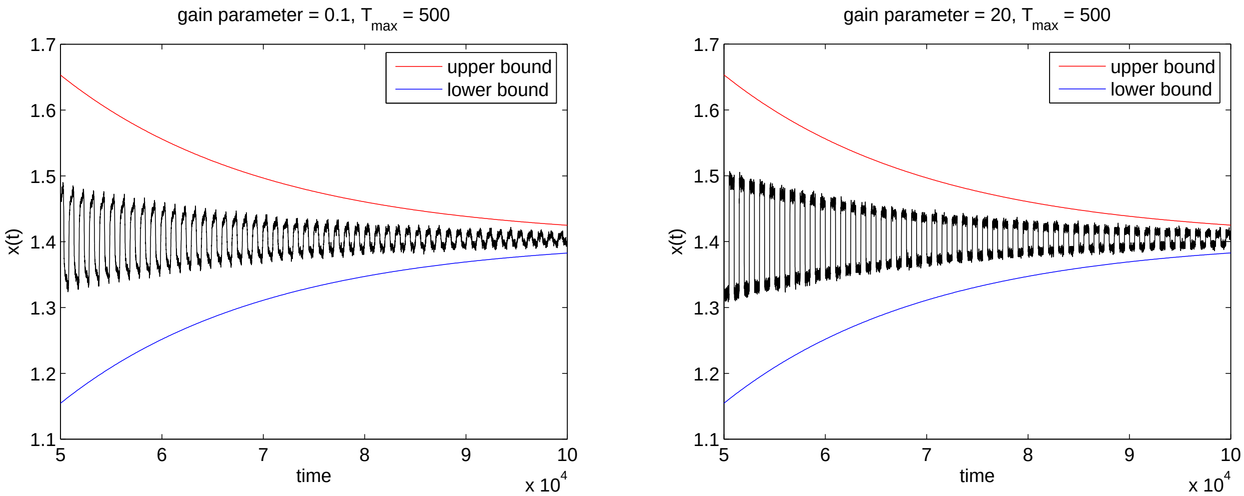

(a) (b)

Fig. 5. Local behavior around the equilibrium. (a) $\kappa_i = 0.1$ and (b) $\kappa_i = 20$.

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APPENDIX I
PROOF OF THEOREM 2

We prove the theorem in two steps. First, we show that the sequence of bounding sets $\tilde{D}_k$ in (26) converges exponentially fast, i.e., there exist some $K_1 > 0$ and $\psi_1 > 0$ such that, for all $k \geq 1$,

$$1 - \overline{\beta}^{\sigma^{2k}} \leq K_1 \exp(-\psi_1 \cdot 2k) \quad \text{and} \quad \overline{\alpha}^{\sigma^{2k}} - 1 \leq K_1 \exp(-\psi_1 \cdot 2k) \tag{33}$$

and

$$1 - \overline{\alpha}^{\sigma^{2k-1}} \leq K_1 \exp(-\psi_1(2k-1)) \quad \text{and} \quad \overline{\beta}^{\sigma^{2k-1}} - 1 \leq K_1 \exp(-\psi_1(2k-1)) \ . \tag{34}$$

Then, we show that there exists a finite positive integer M such that, for all $k \geq 0$, if $\overline{x}_n := (x_{n,1}, \ldots, x_{n,N}) \in \tilde{D}_k, n = 0, -1, \cdots, -\alpha \cdot T_{max}$, then $\overline{x}_n \in \tilde{D}_{k+1}$ for all $n \geq M$. In other words, after a finite number of periods $\overline{x}_n$ converges to $\tilde{D}_{k+1}$ from $\tilde{D}_k$. These two combined satisfy (27) with $K = K_1 \cdot \exp(\psi_1)$ and $\psi(\delta) = \frac{\psi_1}{M + \alpha T_{max}}$.

Lemma 1: Suppose that $0 < \psi_1 \leq \log(-\sigma^{-1})$ and $K_1 = \exp(\psi_1) \cdot (\exp(\overline{\alpha}^{-\sigma^{-1}} - 1) - 1) = \frac{1}{-\sigma}(\exp(\overline{\alpha}^{-\sigma^{-1}} - 1) - 1)$. Then, (33) and (34) hold.

Proof: We first prove (33). Define $K^{'} = \exp(\overline{\alpha}^{-\sigma^{-1}} - 1) - 1 = K_1 \cdot \exp(-\psi_1) < K_1$. Let us first prove the second condition in (33) holds. Add one to both sides and then take the natural logarithm $\log(\cdot)$ of both sides:

$$\begin{aligned}
\log(\overline{\alpha})\sigma^{2k} &= \log(\overline{\alpha}) \exp(-2k \log(-\sigma^{-1})) \\
&\leq (\overline{\alpha} - 1) \exp(-2k \log(-\sigma^{-1})) \\
&\leq \log(1 + (\exp(\overline{\alpha} - 1) - 1) \exp(-2k \log(-\sigma^{-1}))) \\
&\leq \log(1 + K^{'} \exp(-2k\psi_1)) \\
&\leq \log(1 + K_1 \exp(-2k\psi_1)) \ ,
\end{aligned}$$

where the first inequality follows from $\log(1 + x) \leq x$, and the second inequality can be seen from Fig. 6 with $K^{''} = \exp(\overline{\alpha} - 1) - 1 < K^{'}$ and the fact that $\log(-\sigma^{-1}) > 0$ because $-\sigma^{-1} > 1$.

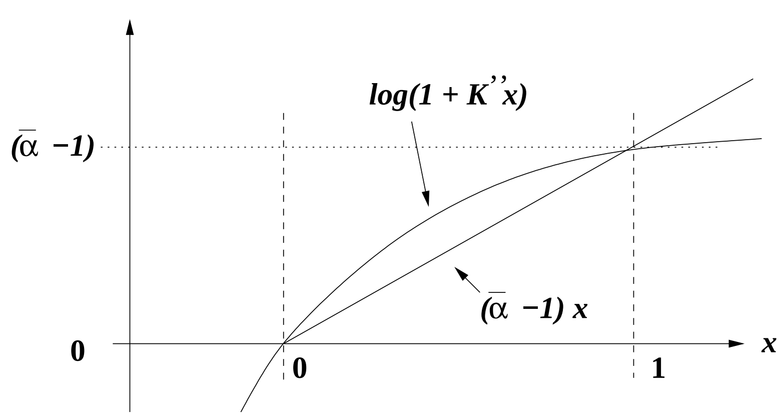

Fig. 6. Proof of Lemma 1.

Similarly, in order to show that the first condition in (33) holds, we show that

$$\overline{\beta}^{\sigma^{2k}} \geq 1 - K_1 \exp(-2k\psi_1) \ . \tag{35}$$

If we take the natural logarithm of the left-hand side,

$$\begin{aligned}\sigma^{2k} \log(\overline{\beta}) &= \log(\overline{\beta}) \exp(-2k \log(-\sigma^{-1})) \\ &= -\log(\overline{\beta}^{-1}) \exp(-2k \log(-\sigma^{-1})) \ ,\end{aligned}$$

and the right-hand side yields

$$\log(1 - K_1 \exp(-2k\psi_1)) = -\log\left((1 - K_1 \exp(-2k\psi_1))^{-1}\right) \ .$$

Hence, it suffices to show that

$$\log(\overline{\beta}^{-1}) \exp(-2k \log(-\sigma^{-1})) \leq \log\left((1 - K_1 \exp(-2k\psi_1))^{-1}\right) \ . \tag{36}$$

We can use a series of inequalities to show (36) as follows.

$$\begin{aligned}\log(\overline{\beta}^{-1}) \exp(-2k \log(-\sigma^{-1})) &\leq (\overline{\beta}^{-1} - 1) \exp(-2k \log(-\sigma^{-1})) \\ &\leq \log\left(1 + (\exp(\overline{\beta}^{-1} - 1) - 1) \exp(-2k \log(-\sigma^{-1}))\right) \\ &\leq \log(1 + K^{'} \exp(-2k\psi_1)) \\ &\leq \log(1 + K_1 \exp(-2k\psi_1)) \\ &\leq \log\left((1 - K_1 \exp(-2k\psi_1))^{-1}\right)\end{aligned}$$

where the second inequality follows from Fig. 6 with $K^{''} = \exp(\overline{\beta}^{-1} - 1) - 1$ and $\overline{\alpha}$ replaced by $\overline{\beta}^{-1}$, and the third inequality is a consequence of $\overline{\beta}^{-1} < (\overline{\alpha}^{\sigma^{-1}})^{-1} = \overline{\alpha}^{-\sigma^{-1}}$ from (25).

Conditions in (34) follow from the fact that $\overline{\beta}^{\sigma^{2k-2}} < \overline{\alpha}^{\sigma^{2k-1}} < \overline{\beta}^{\sigma^{2k-1}} < \overline{\alpha}^{\sigma^{2k-2}}$, which is a consequence of Lemma 2 in [10]. Here we only prove the second condition in (34). The first condition can be proved similarly. First, note that $\overline{\beta}^{\sigma^{2k-1}} < (\overline{\alpha}^{\sigma^{-1}})^{\sigma^{2k-1}} = \overline{\alpha}^{\sigma^{-1} \cdot \sigma^{2k-1}} = \overline{\alpha}^{\sigma^{2k-2}}$. We add one to the left-hand side of the second condition in (34) and then take the natural logarithm:

$$\begin{aligned}\log(\overline{\beta})\sigma^{2k-1} &< \log(\overline{\alpha})\sigma^{2k-2} \\ &= \log(\overline{\alpha})\exp(-2(k-1)\log(\sigma^{-1})) \\ &\leq (\overline{\alpha}-1)\exp(-2(k-1)\log(\sigma^{-1})) \\ &\leq \log\left(1+(\exp(\overline{\alpha}-1)-1)\exp\left(-2(k-1)\log(\sigma^{-1})\right)\right) \\ &\leq \log\left(1+K^{'}\exp(-2(k-1)\psi_1)\right) \\ &= \log\left(1+K^{'}\exp(\psi)\exp(-(2k-1)\psi_1)\right) \\ &= \log\left(1+K_1\exp(-(2k-1)\psi_1)\right) \ .\end{aligned}$$

Hence, the condition is satisfied. ■

Now in order to complete the proof of the theorem, we show that there exists a finite positive integer M such that, for all $k \geq 0$, if $\overline{x}_n \in \tilde{D}_k, n = 0, -1, \cdots, -\alpha \cdot T_{max}$, then $\overline{x}_n \in \tilde{D}_{k+1}$ for all $n \geq M$. First, note from (20) and Lemma 2 in [10] that if $\overline{y}_n \in D_k$ ($\overline{x}_n \in \tilde{D}_k$) and $y_{n,i} > \sup\ D_{k+1,i}$ ($x_{n,i} < \inf\ \tilde{D}_{k+1,i}$), then $y_{n+1,i} \geq \inf\ D_{k+1,i}$ ($x_{n+1,i} \leq \sup\ \tilde{D}_{k+1,i}$), and similarly, if $\overline{y}_n \in D_k$ and $y_{n,i} < \inf\ D_{k+1,i}$ ($x_{n,i} > \sup\ \tilde{D}_{k+1,i}$), then $y_{n+1,i} \leq \sup\ D_{k+1,i}$ ($x_{n+1,i} \geq \inf\ \tilde{D}_{k+1,i}$). We prove the existence of such M in two steps:

(i) Suppose that $\overline{y}_n \in D_k$, $n = 0, -1, \ldots, -\alpha \cdot T_{max}$. If $y_{0,i} \notin D_{k+1,i} := \text{proj}_i(D_{k+1})$, the number of steps it takes for $y_{n,i}$ to get inside $D_{k+1,i}$ is upper bounded by some finite $M(k)$ for all $i \in \mathcal{I}$, i.e.,

$$\inf\{n \geq 1 \mid \overline{y}_n \in D_{k+1}\} \leq M(k) < \infty \ .$$

(ii) There exists $M < \infty$ such that $\sup\ \{M(k), k \in \{1, 2, \ldots\}\} \leq M$.

We first prove (i). We assume that k is even. The case of odd k can be handled similarly. We first consider the case $y_{0,i} < \inf\ D_{k+1,i}$, i.e., $x_{0,i} > \sup\ \tilde{D}_{k+1,i}$. In order to prove (i), we show that if $y_{n,i} < \inf\ D_{k+1,i}$ (which implies $y_{n^{'},i} < \inf\ D_{k+1,i}$ for all $n^{'} = 0, \ldots, n$ by component-wise invariance property), then there exists some constant $\varsigma > 0$ such that $\check{\mathbf{F}}_i(\tilde{Y}_n) - y_{n,i} > \check{\mathbf{F}}_i(\tilde{Y}_n) - \overline{\beta}^{-\sigma^{k+1}a_i} y_i^\star > \varsigma \cdot y_i^\star$.

The existence of such a constant ς can be shown as follows. Let

$$\begin{aligned}
\tilde{x}^n_{(i,l)} &= (x_{n-\alpha(\Gamma^i_{K(i,l)}+\Gamma_1^{j,K(j,l)}),j},\ j \in \mathcal{I})\ ,\ l \in r_i \\
\tilde{X}^i_n &= \left(\overline{x}_{n-\alpha\cdot T_i}, \tilde{x}^n_{(i,l_{(i,1)})}, \ldots, \tilde{x}^n_{(i,l_{(i,R_i)})}\right) \\
\tilde{X}_n &= \left(\tilde{X}^1_n, \ldots, \tilde{X}^N_n\right)
\end{aligned} \tag{37}$$

Define

$$\begin{aligned}
\check{\mathbf{F}}^x_i(\tilde{X}_n) &= (x_{n-\alpha\cdot T_i,i})^{-1/a_i} \left(\sum_{l \in r_i} \left(\frac{\sum_{j\in I_l} x_{n-\alpha\cdot\Gamma^i_{K(i,l)} - \Lambda_1^{j,K(j,l)},j}}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&= g_i^{-1}(\check{\mathbf{F}}_i(\tilde{Y}_n))\ .
\end{aligned}$$

Then,

$$\begin{aligned}
\check{\mathbf{F}}^x_i(\tilde{X}_n) &= (x_{n-\alpha\cdot T_i,i})^{-1/a_i} \left(\sum_{l \in r_i} \left(\frac{\sum_{j\in I_l} x_{n-\alpha\cdot\Gamma^i_{K(i,l)} - \Lambda_1^{j,K(j,l)},j}}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&\le (\overline{\beta}^{\sigma^{k+1}} x_i^\star)^{-1/a_i} \left(\sum_{l \in r_i} \left(\frac{\sum_{j\in I_l} x_{n-\alpha\cdot\Gamma^i_{K(i,l)} - \Lambda_1^{j,K(j,l)},j}}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&\qquad\qquad\qquad\qquad \text{(because } \sup\ \tilde{D}_{k+1,i} < x_{n-\alpha\cdot T_i,i}\text{)} \\
&\le (\overline{\beta}^{\sigma^{k+1}} x_i^\star)^{-1/a_i} \left(\sum_{l \in r_i} \left(\frac{\sum_{j\in I_l} \overline{\beta}^{\sigma^k} x_j^\star}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&\qquad\qquad\qquad\qquad \text{(because } x_{n-\alpha\cdot\Gamma^i_{K(i,l)} - \Lambda_1^{j,K(j,l)},i} \ge \inf\ \tilde{D}_{k,j}\text{)} \\
&\le (\overline{\beta}^{\sigma^{k+1}} x_i^\star)^{-1/a_i} \left(\sum_{l \in r_i} \left(\overline{\beta}^{\sigma^k}\right)^{b^i_{max}} \left(\frac{\sum_{j\in I_l} x_j^\star}{C_l} \right)^{b_l} \right)^{-1/a_i} \quad \text{(because } \overline{\beta}^{\sigma^k} < 1\text{)} \\
&= \overline{\beta}^{-\sigma^{k+1}/a_i} (x_i^\star)^{-1/a_i} \left(\overline{\beta}^{\sigma^k}\right)^{-b^i_{max}/a_i} \left(\sum_{l \in r_i} \left(\frac{\sum_{j\in I_l} x_j^\star}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&= \overline{\beta}^{-\sigma^{k+1}/a_i} \cdot \overline{\beta}^{-\sigma^k \cdot b^i_{max}/a_i} x_i^\star \\
&< (\overline{\beta}^{-1})^{\sigma^k b^i_{max}/a_i} x_i^\star \qquad\qquad \text{(because } \overline{\beta}^{-\sigma^{k+1}/a_i} < 1\text{)} \\
&< (\overline{\beta}^{-1})^{\sigma^k(-\sigma - \frac{1}{a_i})} x_i^\star \qquad\qquad \text{(because } -1 < \sigma < -\tfrac{b^i_{max}+1}{a_i}\text{)}
\end{aligned}$$

$$= \overline{\beta}^{\sigma^k(\sigma+\frac{1}{a_i})} x_i^\star$$
$$= \overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i} x_i^\star$$

where the first three inequalities follow from the monotonicity of the map $\check{\mathbf{F}}_i^x(\cdot)$, the assumption that $x_{n,i} > \sup\ \tilde{D}_{k+1,i} = \overline{\beta}^{\sigma^{k+1}} x_i^\star$, and $\overline{\beta} < 1$. Note that $\overline{\beta}^{\sigma^k/a_i} < 1$ because $\overline{\beta} < 1$. Therefore,

$$g_i(\check{\mathbf{F}}_i^x(\tilde{X}_n)) - g_i(\overline{\beta}^{\sigma^{k+1}} x_i^\star) > g_i(\overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i} x_i^\star) - g_i(\overline{\beta}^{\sigma^{k+1}} x_i^\star) \tag{38}$$

and

$$\inf\{n \geq 1 \mid x_{n,i} \in \tilde{D}_{k+1,i}\} \leq \frac{g_i(\overline{\beta}^{\sigma^{k+1}} x_i^\star) - g_i(\overline{\alpha}^{\sigma^k} x_i^\star)}{\delta \cdot \varepsilon \cdot \left(g_i(\overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i} x_i^\star) - g_i(\overline{\beta}^{\sigma^{k+1}} x_i^\star)\right)} \tag{39}$$

where ε is some positive constant that satisfies $\inf_{\overline{x} \in \tilde{D}_0} \mathbf{K}_{ii}(\overline{g}(\overline{x})) \geq \varepsilon$, and the inequality follows from (38). The existence of such ε is guaranteed because $\tilde{D}_0$ is a compact set. One can show from the definition $g_i(x) = x^{-a_i}$ that

$$\begin{aligned} \frac{g_i(\overline{\beta}^{\sigma^{k+1}} x_i^\star) - g_i(\overline{\alpha}^{\sigma^k} x_i^\star)}{g_i(\overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i} x_i^\star) - g_i(\overline{\beta}^{\sigma^{k+1}} x_i^\star)} &\leq \frac{\overline{\alpha}^{\sigma^k} x_i^\star - \overline{\beta}^{\sigma^{k+1}} x_i^\star}{\overline{\beta}^{\sigma^{k+1}} x_i^\star - \overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i} x_i^\star} \\ &= \frac{\overline{\alpha}^{\sigma^k} - \overline{\beta}^{\sigma^{k+1}}}{\overline{\beta}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i}} \ . \end{aligned} \tag{40}$$

Thus, from (39) and (40)

$$\begin{aligned} \inf\{n \geq 1 \mid x_{n,i} \in \tilde{D}_{k+1,i}\} &\leq \frac{\overline{\alpha}^{\sigma^k} - \overline{\beta}^{\sigma^{k+1}}}{\delta \cdot \varepsilon \cdot (\overline{\beta}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^{k+1}} \overline{\beta}^{\sigma^k/a_i})} \\ &= \frac{\overline{\alpha}^{\sigma^k} - \overline{\beta}^{\sigma^{k+1}}}{\delta \cdot \varepsilon \cdot \overline{\beta}^{\sigma^{k+1}} (1 - \overline{\beta}^{\sigma^k/a_i})} \\ &\leq \frac{\overline{\alpha}^{\sigma^k} - (\overline{\alpha}^{\sigma})^{\sigma^{k+1}}}{\delta \cdot \varepsilon \cdot (\overline{\alpha}^{\sigma})^{\sigma^{k+1}} (1 - (\overline{\alpha}^{\sigma})^{\sigma^k/a_i})} \\ &= \frac{\overline{\alpha}^{\sigma^k} - \overline{\alpha}^{\sigma^{k+2}}}{\delta \cdot \varepsilon \cdot \overline{\alpha}^{\sigma^{k+2}} (1 - \overline{\alpha}^{\sigma^{k+1}/a_i})} \\ &=: M_i^u(k) \ , \end{aligned} \tag{41}$$

where the second inequality follows from the assumption $\overline{\beta} < \overline{\alpha}^{\sigma} < 1$.

We now turn to the other case $y_{0,i} > \sup\ D_{k+1,i}$, i.e., $x_{0,i} < \inf\ \tilde{D}_{k+1,i}$. Similarly as in the first case we show that if $y_{n,i} > \sup\ D_{k+1,i}$ (which implies $y_{n',i} > \sup\ D_{k+1,i}$ for all $n' = 0, \ldots, n$), then there exists some constant $\varsigma > 0$ such that $y_{n,i} - \check{\mathbf{F}}_i(\tilde{Y}_n) > g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star) - \check{\mathbf{F}}_i(\tilde{Y}_n) = \overline{\alpha}^{-\sigma^{k+1}a_i} y_i^\star - \check{\mathbf{F}}_i(\tilde{Y}_n) > \varsigma \cdot y_i^\star$.

$$\begin{aligned}
\check{\mathbf{F}}_i^x(\tilde{X}_n) &= (x_{n-\alpha\cdot T_i,i})^{-1/a_i} \left(\sum_{l\in r_i} \left(\frac{\sum_{j\in I_l} x_{n-\alpha\cdot\Gamma^i_{K(i,l)} - \Lambda_1^{j,K(j,l)},j}}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&\geq (\overline{\alpha}^{\sigma^{k+1}} x_i^\star)^{-1/a_i} \left(\sum_{l\in r_i} \left(\frac{\sum_{j\in I_l} \overline{\alpha}^{\sigma^k} x_j^\star}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&\geq (\overline{\alpha}^{\sigma^{k+1}} x_i^\star)^{-1/a_i} \left(\sum_{l\in r_i} \left(\overline{\alpha}^{\sigma^k}\right)^{b^i_{max}} \left(\frac{\sum_{j\in I_l} x_j^\star}{C_l} \right)^{b_l} \right)^{-1/a_i} \\
&= \overline{\alpha}^{-\sigma^{k+1}/a_i} \cdot \overline{\alpha}^{-\sigma^k\cdot b^i_{max}/a_i} x_i^\star \\
&> (\overline{\alpha}^{-1})^{\sigma^k b^i_{max}/a_i} x_i^\star && \text{(because } \overline{\alpha}^{-\sigma^{k+1}/a_i} > 1\text{)} \\
&> (\overline{\alpha}^{-1})^{\sigma^k(-\sigma - \frac{1}{a_i})} x_i^\star && \text{(because } -1 < \sigma < -\tfrac{b^i_{max}+1}{a_i}\text{)} \\
&= \overline{\alpha}^{\sigma^k(\sigma + \frac{1}{a_i})} x_i^\star \\
&= \overline{\alpha}^{\sigma^{k+1}} \overline{\alpha}^{\sigma^k/a_i} x_i^\star
\end{aligned}$$

where the first two inequalities follow from the monotonicity of the map $\check{\mathbf{F}}_i^x(\cdot)$, the assumption that $x_{n,i} < \inf\ \tilde{D}_{k+1,i} = \overline{\alpha}^{\sigma^{k+1}} x_i^\star$, and $\overline{\alpha} > 1$. Note that $\overline{\alpha}^{\sigma^k/a_i} > 1$ because $\overline{\alpha} > 1$. Hence,

$$g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star) - g_i(\check{\mathbf{F}}_i^x(\tilde{X}_n)) > g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}} \overline{\alpha}^{\sigma^k/a_i} x_i^\star) \tag{42}$$

and

$$\inf\{n \geq 1 \mid x_{n,i} \in \tilde{D}_{k+1,i}\} \leq \frac{g_i(\overline{\beta}^{\sigma^k} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star)}{\delta \cdot \varepsilon \cdot \left(g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}} \overline{\alpha}^{\sigma^k/a_i} x_i^\star) \right)}\ . \tag{43}$$

Using the definition of $g_i(x) = x^{-a_i}$,

$$\begin{aligned}
\frac{g_i(\overline{\beta}^{\sigma^k} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star)}{g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}} \overline{\alpha}^{\sigma^k/a_i} x_i^\star)} &\geq \frac{\overline{\alpha}^{\sigma^{k+1}} x_i^\star - \overline{\beta}^{\sigma^k} x_i^\star}{\overline{\alpha}^{\sigma^{k+1}} \overline{\alpha}^{\sigma^k/a_i} x_i^\star - \overline{\alpha}^{\sigma^{k+1}} x_i^\star} \\
&= \frac{\overline{\alpha}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^k}}{\overline{\alpha}^{\sigma^{k+1}} \overline{\alpha}^{\sigma^k/a_i} - \overline{\alpha}^{\sigma^{k+1}}}\ ,
\end{aligned} \tag{44}$$

and the ratio

$$\Phi(k) = \frac{(g_i(\overline{\beta}^{\sigma^k} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star))/(g_i(\overline{\alpha}^{\sigma^{k+1}} x_i^\star) - g_i(\overline{\alpha}^{\sigma^{k+1}}\overline{\alpha}^{\sigma^k/a_i} x_i^\star))}{(\overline{\alpha}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^k})/(\overline{\alpha}^{\sigma^{k+1}}\overline{\alpha}^{\sigma^k/a_i} - \overline{\alpha}^{\sigma^{k+1}})} \tag{45}$$
$$= \frac{(\overline{\beta}^{-\sigma^k a_i} y_i^\star - \overline{\alpha}^{-\sigma^{k+1}a_i} y_i^\star)/(\overline{\alpha}^{-\sigma^{k+1}a_i} y_i^\star - \overline{\alpha}^{-\sigma^{k+1}a_i}\overline{\alpha}^{-\sigma^k} y_i^\star)}{(\overline{\alpha}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^k})/(\overline{\alpha}^{\sigma^{k+1}}\overline{\alpha}^{\sigma^k/a_i} - \overline{\alpha}^{\sigma^{k+1}})}$$
$$= \frac{(\overline{\beta}^{-\sigma^k a_i} - \overline{\alpha}^{-\sigma^{k+1}a_i})/(\overline{\alpha}^{-\sigma^{k+1}a_i} - \overline{\alpha}^{-\sigma^{k+1}a_i}\overline{\alpha}^{-\sigma^k})}{(\overline{\alpha}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^k})/(\overline{\alpha}^{\sigma^{k+1}}\overline{\alpha}^{\sigma^k/a_i} - \overline{\alpha}^{\sigma^{k+1}})}$$

is bounded by some finite constant $\Phi^\star$ for all $k = 1, 2, \ldots$, with $\lim_{k\to\infty} \Phi(k) = 1$. Thus, from (43) - (45)

$$\begin{aligned}
\inf\{n \geq 1 \mid x_{n,i} \in \tilde{D}_{k+1,i}\} &\leq \Phi^\star \frac{\overline{\alpha}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^k}}{\delta \cdot \varepsilon \cdot (\overline{\alpha}^{\sigma^{k+1}}\overline{\alpha}^{\sigma^k/a_i} - \overline{\alpha}^{\sigma^{k+1}})} \\
&= \Phi^\star \frac{\overline{\alpha}^{\sigma^{k+1}} - \overline{\beta}^{\sigma^k}}{\delta \cdot \varepsilon \cdot \overline{\alpha}^{\sigma^{k+1}}(\overline{\alpha}^{\sigma^k/a_i} - 1)} \\
&\leq \Phi^\star \frac{\overline{\alpha}^{\sigma^{k+1}} - \left(\overline{\alpha}^{\sigma^{-1}}\right)^{\sigma^k}}{\delta \cdot \varepsilon \cdot \overline{\alpha}^{\sigma^{k+1}}(\overline{\alpha}^{\sigma^k/a_i} - 1)} \\
&= \Phi^\star \frac{\overline{\alpha}^{\sigma^{k+1}} - \overline{\alpha}^{\sigma^{k-1}}}{\delta \cdot \varepsilon \cdot \overline{\alpha}^{\sigma^{k+1}}(\overline{\alpha}^{\sigma^k/a_i} - 1)} \qquad (46) \\
&=: M_i^l(k) \ ,
\end{aligned}$$

where the second inequality follows from the assumption $\overline{\alpha}^{\sigma^{-1}} < \overline{\beta} < 1$, and ε is some positive constant that satisfies $\inf_{\overline{x}\in\tilde{D}_0} \mathbf{K}_{ii}(\overline{g}(\overline{x})) \geq \varepsilon$ as before. Now, letting $M(k) = \max(\max_{i\in\mathcal{I}} M_i^l(k)$, $\max_{i\in\mathcal{I}} M_i^u(k))$ satisfies (i).

Claim (ii) can be established by showing that $M_i^u(k)$ $(M_i^l(k))$ defined in (41) (resp. (46)) is monotonically decreasing (resp. increasing) in k and has a well defined *finite* limit as $k \to \infty$ using L'Hôpital's rule. The limits of (41) and (46) are given by $-\frac{a_i(1-\sigma^2)}{\delta\cdot\varepsilon\cdot\sigma}$ and $-\Phi^\star\frac{a_i(1-\sigma^2)}{\delta\cdot\varepsilon\cdot\sigma}$, respectively. This guarantees that there exists some finite M such that $\sup\{M(k); k = 1, 2, \ldots\} \leq M$, and the theorem follows.

APPENDIX II

PROOF OF THEOREM 3

Corollary 3.2 of [5] states that the delay differential system in (7) with a single resource and a single flow will either converge to $y^\star$ or oscillate around $y^\star$. Define

$$\underline{m} := \liminf_{t\to\infty} y(t) \quad \text{and} \quad \overline{m} := \limsup_{t\to\infty} y(t) \ .$$

Let $y_t : \mathbb{R}_+ \to \mathcal{C}([-T,0],\mathbb{R}_+)$, where $y_t(s) = y(t+s)$, $s \in [-T,0]$. Define $\Omega(y) := \cap_{t\geq 0}\{y_r, r \geq t\} \subset \mathcal{C}([-T,0],\mathbb{R}_+)$. Then, for every $\phi \in \Omega(y)$ there exists a solution $y(s;\phi) : \mathbb{R} \to \mathbb{R}_+$ such that $\phi(s) = y(s;\phi)$, $s \in [-T,0]$. Further, for all $\phi \in \Omega(y)$ we have

$$\liminf_{t\to\infty} y(t) = \underline{m} \leq y(s,\phi) \leq \overline{m} = \limsup_{t\to\infty} y(t) \ .$$

In addition,

$$\overline{m} = y(t_1;\psi) \quad \text{and} \quad \underline{m} = y(t_2;\xi) \tag{47}$$

for some $\psi, \xi \in \Omega(y)$ and $t_1, t_2 > 0$. Lemma 3.3. of [5] tells us that if $y(t_1) = \overline{m}$, then $y(t_1 - T) \leq y^\star$. Similarly, if $y(t_2) = \underline{m}$, then $y(t_2 - T) \geq y^\star$. Therefore, without loss of generality we can assume that

$$y(0;\psi) = y(0;\xi) = y^\star \quad \text{and} \quad t_i \leq T, i = 1,2 \tag{48}$$

with $y(t;\psi) \in (y^\star, \overline{m})$ for all $t \in (0,t_1)$ and $y(t;\xi) \in (\underline{m}, y^\star)$ for all $t \in (0,t_2)$.

First, consider $x(t) = x(t;\psi_x)$ constructed by (4), where $\psi_x(s) = g^{-1}(\psi(s))$ for all $s \in [-T,0]$. Let $z(t)$ be the solution to the following initial value problem: $z(0) = x^\star$ and

$$\frac{d}{dt} z(t) = \kappa \cdot z(t) U'(z(t)) \ .$$

Then, since $(x(t)U'(x(t)) - x(t-T)p(x(t-T))) \leq x(t)U'(x(t))$, we have $x(t) \leq z(t), t \geq 0$ [14, Theorem 5.III]. Substituting $U'(x) = x^{-(1+a)}$, after a little algebra, one can show that

$$z(t) = \left(x^{\star 1+a} + (1+a)\kappa \cdot t\right)^{1/(1+a)}$$

and

$$z(T) = \left(x^{\star 1+a} + (1+a)\kappa \cdot T\right)^{1/(1+a)} \geq z(t_1) \geq x(t_1) = \limsup_{t\to\infty} x(t;\psi_x) = g^{-1}(\liminf_{t\to\infty} y(t;\psi)) \ .$$

Therefore, $z(T)$ provides an upper bound on $\limsup_{t\to\infty} x(t;\psi_x)$. Furthermore, since $g(x)$ is a decreasing function of x from Assumption 2, this tells us

$$\liminf_{t\to\infty} y(t;\psi) \geq g(z(T)) =: y_* \ .$$

Let $v(t)$ be the solution to the following initial value problem: $v(0) = y^\star$ and

$$\frac{d}{dt}v(t) = \mathbf{K}(v(t))\left(\mathbf{F}(y_*) - v(t)\right) \ ,$$

where $\mathbf{F}(y) = C^{-b}y^{-\frac{1+b}{a}}$. Then, since $\mathbf{F}(y_*) \geq \mathbf{F}(y(t-T))$ for all $t \geq 0$, we have $(\mathbf{F}(y(t-T)) - y(t)) \leq (\mathbf{F}(y_*) - y(t))$ and, hence, $v(t) \geq y(t)$ for all $t \in (0, t_1)$ [14, Theorem 5.III]. Therefore, $v(T)(\geq y^\star)$ provides an upper bound on $\limsup_{t\to\infty} y(t)$. Now let

$$\Theta(T,\kappa) = \frac{\mathbf{K}(v(T)) \cdot T}{\kappa} = a \cdot T \cdot v(T)^{1+\frac{1}{a}} \ .$$

Define a map

$$\begin{aligned}
\mathbf{G}(y) \ &:= \ (1 - \exp(-\kappa \cdot \Theta(T,\kappa)))\mathbf{F}(y) + [y^\star - (1 - \exp(-\kappa \cdot \Theta(T,\kappa)))\mathbf{F}(y^\star)] \\
&= \ (1 - \exp(-\kappa \cdot \Theta(T,\kappa)))\mathbf{F}(y) + y^\star \cdot \exp(-\kappa \cdot \Theta(T,\kappa)),
\end{aligned}$$

where $y^\star = C^{-\frac{ab}{1+a+b}}$.

Lemma 2: The map $\mathbf{G}(y)$ has a globally attracting fixed point $y^\star$ under the condition in Theorem 3.

Proof: First, note that the first derivative $\mathbf{G}'(y)$ evaluated at $y^\star$ is given by

$$\mathbf{G}'(y)|_{y=y^\star} = -\frac{1+b}{a}(1 - \exp(-\kappa \cdot \Theta(T,\kappa))) \ .$$

Therefore, the condition $\frac{1+b}{a}(1 - \exp(-\kappa \cdot \Theta(T,\kappa)) < 1$ is equivalent to the local stability of the map $\mathbf{G}$ around the fixed point $y^\star$.

The most general condition of Sharkovsky cycle theorem states that the fixed point $y^\star$ is globally attracting if (i) the second iterate $\mathbf{G}^2(y) := \mathbf{G}(\mathbf{G}(y))$ of the map $\mathbf{G}$ does not have a fixed point in the relevant interval other than $y^\star$, and (ii) $y^\star$ is locally stable. Hence, in order to prove the lemma we only need to show that the second iterate $\mathbf{G}^2(y)$ does not have any other fixed point.

Since the map $\mathbf{G}(y)$ is strictly decreasing in y, if there exist a pair of period two fixed points of $\mathbf{G}$, then one of them must be strictly larger than $y^\star$, while the other is strictly smaller than $y^\star$. We will show that there cannot exist a period two fixed point larger than $y^\star$. A sufficient condition for this is that $\mathbf{G}^{2\prime}(y) < 1$ for all $y \geq y^\star$.

$$\begin{aligned}\mathbf{G}^{2\prime}(y) &= \mathbf{G}'(\mathbf{G}(y))\mathbf{G}'(y) && \text{(by the chain rule)}\\ &= \left(\frac{1+b}{a}C^{-b}(1-\exp(-\kappa\cdot\Theta(T,\kappa)))\right)^2 (y\mathbf{G}(y))^{-\frac{1+a+b}{a}} \ .\end{aligned}$$

Since we know that $\mathbf{G}^{2\prime}(y^\star) = (\mathbf{G}'(y^\star))^2 < 1$, it suffices to prove that $y\mathbf{G}(y) \geq (y^\star)^2$ for all $y > y^\star$. Define $H(y) = y\mathbf{G}(y)$. We will show that $H'(y) > 0$ for all $y > y^\star$.

$$\begin{aligned}H'(y) = &\frac{1}{a}C^{-b\left(\frac{a}{1+a+b}+1\right)}y^{-\frac{1+b}{a}}\\ &\times\left((a-(1+b))C^{\frac{ab}{1+a+b}}(1-\exp(-\kappa\cdot\Theta(T,\kappa))) + aC^b\exp(-\kappa\cdot\Theta(T,\kappa))y^{\frac{1+b}{a}}\right)\end{aligned}$$

Thus, in order to prove that $H'(y) > 0$ for all $y > y^\star$, we need to show that

$$C^b\exp(-\kappa\cdot\Theta(T,\kappa))y^{\frac{1+b}{a}} \geq \left(\frac{1+b}{a}-1\right)C^{\frac{ab}{1+a+b}}(1-\exp(-\kappa\cdot\Theta(T,\kappa))) \ . \tag{49}$$

First, note that the left-hand side of (49) is strictly increasing in y, while the right-hand side is a constant. Hence, the claim will follow if we show that $C^b\exp(-T\cdot\mathbf{K}^\star)y^{\star\frac{1+b}{a}}$ is greater than or equal to the right-hand side of (49).

$$\begin{aligned}C^b y^{\star\frac{1+b}{a}}\exp(-\kappa\cdot\Theta(T,\kappa)) &= C^b\left(C^{-\frac{ab}{1+a+b}}\right)^{\frac{1+b}{a}}\exp(-\kappa\cdot\Theta(T,\kappa))\\ &= C^{\frac{ab}{1+a+b}}\exp(-\kappa\cdot\Theta(T,\kappa))\end{aligned}$$

Thus, we need to show that

$$\exp(-\kappa\cdot\Theta(T,\kappa)) \geq \left(\frac{1+b}{a}-1\right)(1-\exp(-\kappa\cdot\Theta(T,\kappa)) \ .$$

This can be shown as follows.

$$\begin{aligned}\left(\frac{1+b}{a}-1\right)(1-\exp(-\kappa\cdot\Theta(T,\kappa))) &= \frac{1+b}{a}(1-\exp(-\kappa\cdot\Theta(T,\kappa))) - 1 + \exp(-\kappa\cdot\Theta(T,\kappa))\\ &< 1 - 1 + \exp(-\kappa\cdot\Theta(T,\kappa))\\ &= \exp(-\kappa\cdot\Theta(T,\kappa)) \ ,\end{aligned}$$

where the inequality follows from the condition in the theorem. ■

We now continue with the proof of Theorem 3. Take an arbitrary solution $y(t)$ of (7) and let

$$0 < \underline{m} = \liminf_{t\to\infty} y(t) \le y(t) \le \overline{m} = \limsup_{t\to\infty} y(t) \ .$$

Suppose that ψ and ξ be initial functions satisfying (47) and (48). Define

$$\mathbf{F}_+(q(t),\underline{m},\overline{m}) := \mathbf{K}(q(t))\left(\mathbf{F}(\underline{m}) - q(t)\right) \quad \text{and} \quad \mathbf{F}_-(q(t),\underline{m},\overline{m}) := \mathbf{K}(q(t))\left(\mathbf{F}(\overline{m}) - q(t)\right) \ .$$

Let $z(t)$ be the solution to the initial value problem of

(i) $z(0) = y^\star$, and

(ii) $\frac{d}{dt}z(t) = \mathbf{F}_+(z(t),\underline{m},\overline{m})$.

Note that $\Psi := \Psi(\underline{m},\overline{m}) = z(t_1,\underline{m},\overline{m}) \ge \overline{m}$ satisfies[7]

$$\int_{y^\star}^{\Psi} \frac{dz}{\mathbf{F}_+(z,\underline{m},\overline{m})} = t_1 \le T \ .$$

Hence, if we define $\Phi_+ := \Phi_+(\underline{m},\overline{m}) > y^\star$ as the unique solution of

$$\int_{y^\star}^{\Phi_+} \frac{dz}{\mathbf{F}_+(z,\underline{m},\overline{m})} = T \ ,$$

then we have $\Phi_+ \ge \Psi \ge \overline{m}$. Also, $\Phi_+ \le v(T)$ from the construction of $v(T)$ because $y_* \le \underline{m}$. Likewise, an analogous consideration of $y(t;\xi)$ shows that $\Phi_- := \Phi_-(\underline{m},\overline{m}) \le \underline{m}$, where Φ_- is defined by

$$\int_{y^\star}^{\Phi_-} \frac{dz}{\mathbf{F}_-(z,\underline{m},\overline{m})} = T \ .$$

Note that

$$\begin{aligned} T = \int_{y^\star}^{\Phi_+} \frac{dz}{\mathbf{F}_+(z,\underline{m},\overline{m})} &= \int_{y^\star}^{\Phi_+} \frac{dz}{\mathbf{K}(z)(\mathbf{F}(\underline{m}) - z)} \\ \ge \int_{y^\star}^{\Phi_+} \frac{dz}{\mathbf{K}(v(T))(\mathbf{F}(\underline{m}) - z)} \ , \end{aligned}$$

where the inequality follows from the fact that $v(T) \ge \Phi_+$ and $\mathbf{K}(y) = a \cdot \kappa \cdot y^{1+\frac{1}{a}}$ is an increasing function of y. Integrating this last equation, we obtain

$$T \ge \frac{-1}{\mathbf{K}(v(T))} \ln\left(\frac{\Phi_+ - \mathbf{F}(\underline{m})}{y^\star - \mathbf{F}(\underline{m})}\right)$$

[7] The existence of such Ψ can be shown.

This, with $\mathbf{K}(v(T)) = \frac{\kappa \cdot \Theta(T,\kappa)}{T}$, yields

$$\begin{aligned}\Phi_+ &\leq \mathbf{F}(\underline{m}) + \exp(-\kappa \cdot \Theta(T,\kappa))(y^\star - \mathbf{F}(\underline{m})) \\ &= \mathbf{F}(\underline{m})(1 - \exp(-\kappa \cdot \Theta(T,\kappa))) + y^\star \exp(-\kappa \cdot \Theta(T,\kappa)) \\ &= \max_{y \in [\underline{m},\overline{m}]} \mathbf{G}(y) \ . \end{aligned} \tag{50}$$

Following the same steps one can show that

$$\Phi_- \geq \min_{y \in [\underline{m},\overline{m}]} \mathbf{G}(y) \ . \tag{51}$$

Eqs. (50) and (51) imply that $[\underline{m}, \overline{m}] \subset \mathbf{G}([\underline{m}, \overline{m}])$. However, Lemma 2 tells us that the map $\mathbf{G}(y)$ has a globally attracting fixed point $y^\star$ and thus $\mathbf{G}([\underline{m}, \overline{m}]) \subset [\underline{m}, \overline{m}]$. This implies $\underline{m} = y^\star = \overline{m}$, proving that the system is asymptotically globally stable.

TABLE I

DEFINITIONS OF VARIABLES.

Variable	Definition
B_l	buffer size of link $l \in \mathcal{L}$
C_l	capacity of link $l \in \mathcal{L}$
d_l	propagation and transmission delay of link $l \in \mathcal{L}$
$\check{\mathbf{F}}_i(\cdot)$ and $\mathbf{F}_i(\cdot)$	function that gives the total price of flow i based on the current and past willingness to pay of the flows
$\hat{\mathbf{F}}_i(\overline{x})$	function that gives flow i's rate generating its willingness to pay equal to $x_i \cdot \sum_{l \in r_i} p_l(\sum_{j \in I_l} x_j) =: f_i(\overline{x})$, i.e., $g_i(\hat{\mathbf{F}}_i(\overline{x})) = f_i(\overline{x})$
$\check{f}_i(\cdot)$	function that gives the total price of flow i based on the current and past rates of the flows
$g_i(x_i)$	willingness to pay of user i when it receives a rate x_i
I_l	set of flows traversing link l, i.e., $\{i \in \mathcal{I} \| l \in r_i\}$
$\mathbf{K}(\overline{y})$	diagonal state dependent gain matrix of the flows with $\mathbf{K}_{ii}(\overline{y}) = -\kappa_i g_i'(g_i^{-1}(y_i))$
$T^f_{i,l_{i,m}}$	forward delay of user i's packet from sender i to link $l_{i,m}, m = 1, \ldots, R_i$
p_l	price function of link $l \in \mathcal{L}$
T_{max}	an upper bound on the maximum delay experienced by feedback information
T_i	user i's round-trip delay
$T^r_{i,l_{i,m}}$	reverse path delay from resource $l_{i,m}$ to sender i
$U_i(x_i)$	utility of flow i as a function of its rate x_i
$\overline{x}^\star$ and $x^\star$	a unique solution to (3)
$x_i(t)$	flow i's rate at time t ($\overline{x}(t) = (x_i(t), i \in \mathcal{I})$)
$Y(t)$	$\left(Y^1(t), \ldots, Y^N(t)\right)$
$Y^i(t)$	vector containing the current and past willingness to pay of the flows, i.e., $\left(\overline{y}(t - T_i), \tilde{y}_{(i,l_{i,1})}(t), \ldots, \tilde{y}_{(i,l_{i,R_i})}(t)\right)$
$\overline{y}^\star$ and $y^\star$	willingness to pay of flow(s) at the solution $\overline{x}^\star$ or $x^\star$
$y_i(t)$	willingness to pay of user i at time t ($\overline{y}(t) = (y_i(t), i \in \mathcal{I})$)
$\tilde{y}_{(i,l)}(t)$	$\left(y_j(t - T^r_{i,l} - T^f_{j,l}), j \in \mathcal{I}\right)$
Δ	$\max_{i \in \mathcal{I}} \frac{b^i_{max}+1}{a_i}$