2011 spring

ENEE 661 Lecture 1(b)

Frenet-Serret formulas and control systems on Lie groups.

Consider a C^3 curve in $\mathbb{R}^3$, $t \mapsto \gamma(t)$ starting at $t = t_0$ at $\gamma(t_0) = \gamma_0$.

Let $s(t) = \int_{t_0}^{t} \left(\frac{d\gamma}{dt} \cdot \frac{d\gamma}{dt}\right)^{1/2} dt$ denote the length of the curve γ from t_0 to t.

The dot product is the Euclidean inner product.

Then, speed $\frac{ds}{dt} = \|\dot{\gamma}(t)\| = (\dot{\gamma}(t) \cdot \dot{\gamma}(t))^{1/2}$

Hypothesis 1: $\dot{\gamma}(t) \neq 0$ for any $t \geq t_0$ (regular curve)

Then $s(t)$ is strict monotonic function of t and can be inverted in principle to obtain $t = t(s)$. Note $t_0 = t(0)$.

Thus the curve can be reparametrized in terms of s by expressing

$$\gamma = \gamma(t) = \gamma(t(s))$$

Definition 2: We call the above re-parametrization, the arc-length parametrization.

We can write tangent $T(s) \triangleq \frac{d\gamma}{ds}$

$= \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{d\gamma}{dt} \Big/ \frac{ds}{dt}$. Then

$$\|T(s)\| = \left\|\frac{d\gamma}{dt}\right\| \Big/ \left|\frac{ds}{dt}\right| = \frac{ds}{dt} \Big/ \frac{ds}{dt} = 1$$

for all $s \geq 0$.

Thus in the arc length parametrization, the curve γ has unit speed.

So we also refer,
to the arclength parametrization as the
unit speed parametrization.

Observation 3 Changing the (laboratory) coordinate system into a new one by rotation and translation the original curve γ becomes a new curve $\tilde{\gamma}$

$$\tilde{\gamma}(t) = P\gamma(t) + b$$

where $P \in SO(3)$ and $b \in \mathbb{R}^3$.

Since $\dot{\tilde{\gamma}}(t) = P\,\dot{\gamma}(t)$, it follows that arc-length

$$\tilde{s}(t) = \int_0^t \left\| \frac{d\tilde{\gamma}}{dt} \right\| dt$$

$$= \int_0^t \left\| \frac{d\gamma}{dt} \right\| dt$$

$$= s(t)$$

i.e. arc-length is invariant under $SE(3)$.
We seek other invariants.

Observation 4 $T(s) \cdot T(s) \equiv 1$.

Differentiate to obtain

$$T'(s) \cdot T(s) \equiv 0$$

where $'$ denotes $\frac{d}{ds}$.

Definition 5 Curvature $\kappa(s) = \left\| \frac{dT}{ds} \right\| \geq 0$.

It is invariant under $SE(3)$ action $\gamma \mapsto P\gamma + b$

Proposition 6 $\kappa(s) \equiv 0$ on an interval of definition of a curve iff $\gamma(s)$ is a straight line on that interval.

Proof: $(\Rightarrow)$ $\kappa(s) \equiv 0 \iff \left\| \frac{dT}{ds} \right\| \equiv 0$ on an interval

$\iff \frac{dT}{ds} \equiv 0$ on an interval

$\Rightarrow T(s) \equiv \text{constant} = \underline{c}$

$\Leftrightarrow \frac{d\gamma}{ds} = \underline{c}$

$\Rightarrow \gamma(s) = \gamma(0) + s\,\underline{c}$ (straight line)

$(\Leftarrow)$ Trace backward the above steps. $\square$

Definition 7 If $\kappa(s_1) \neq 0$ for a particular s_1 then we can define the unit <u>normal</u> vector

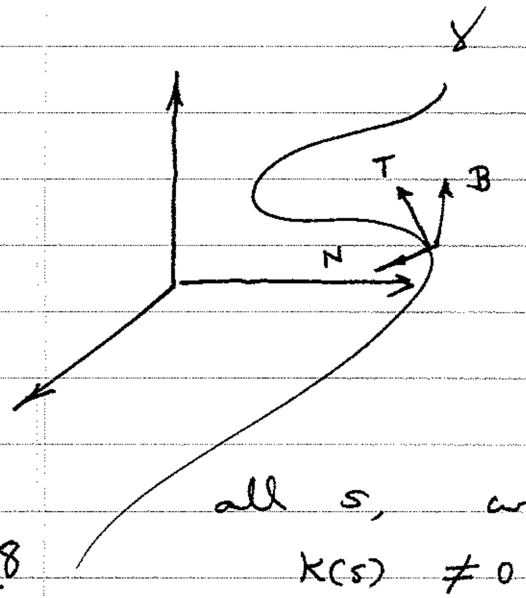
Normal & Binormal

$$N(s_1) = T'(s_1) / \kappa(s_1) .$$

By continuity, such a normal is defined on a neighborhood of s_1. On that neighborhood, one lets

$$B(s) = T(s) \times N(s) \qquad \text{(}\underline{\text{binormal}}\text{)}$$

and thus obtains the orthonormal triad $\{ T(s), N(s), B(s) \}$. We call this the Frenet-Serret frame of the curve.



Recall that this construction works only on a neighbor-hood of s_1 where $\kappa(s_1) \neq 0$, to avoid division by zero in the definition of N.

To make this work for all s, we need,

Hypothesis 8 $\quad \kappa(s) \neq 0 \quad \forall s \quad$ (nondegeneracy)

This holds generically. Under this hypothesis one can derive a set of differential equations to evolve the triad $\{T(s), N(s), B(s)\}$.

Let $F \triangleq [F_1(s) \;\; F_2(s) \;\; F_3(s)] \triangleq [T(s) \;\; N(s) \;\; B(s)]$

Clearly $F^T F \equiv \mathbb{1}$ and $\det(F) = +1$ since the triad $\{T, N, B\}$ is right handed.

Thus $s \mapsto F(s)$ defines a curve in $SO(3)$.

See homework assignment 1. $\leftarrow$ $\boxed{\text{We know (Lect 3, p. 9, Example 9)}}$ that F is generated by a skew symmetric (s-dependent) matrix $\hat{\Omega}$:

$$\frac{dF(s)}{ds} = F(s) \cdot \hat{\Omega}(s)$$

where $\hat{\Omega} + \hat{\Omega}^T \equiv 0$.

The structure of Ω is easy to work out. Write $\quad \hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$

$$\frac{dF_1}{ds} = T(s) = F(s) \cdot \text{first column of } \Omega$$

$$= T(s) \cdot 0 + N(s)\,\Omega_3(s) + B(s)\,(-\Omega_2)$$

$$= N(s)\ \kappa(s) \qquad \text{by definition of } N$$

$$\Rightarrow \quad \Omega_3 = \kappa \quad \text{and} \quad \Omega_2 \equiv 0 .$$

$$\frac{dF_2}{ds} = N(s) = F(s) \cdot \text{second column of } \Omega$$

$$= T(s)\,(-\Omega_3) + B(s)\,(\Omega_1)$$

$$= -\kappa\, T(s) + \tau\, B(s)$$

where we define $\tau(s) = \Omega_1(s)$ (torsion)

$$\frac{dF_3}{ds} = \frac{dB}{ds} = F(s) \cdot 3^{rd} \text{ column of } \Omega$$

$$= -\tau(s)\, N(s)$$

The last equation also tells us

$$\tau(s) = -\frac{dB}{ds} \cdot N(s)$$

We can take this to be the definition of torsion.

Thus

$$\frac{d}{ds}\begin{bmatrix} T(s) & N(s) & B(s) \end{bmatrix} = \begin{bmatrix} T(s) & N(s) & B(s) \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}$$

These are the <u>Frenet-Serret equations</u>.

Given a <u>program</u> of curvature $\kappa(s)$ and torsion $\tau(s)$ we can integrate the above system of equations starting from an initial frame and compute the curve γ by

$$\gamma(s) = \gamma(0) + \int_0^s T(\sigma)\, d\sigma .$$

<u>Proposition</u> 9 A curve is planar iff $\tau(s) \equiv 0$.

<u>Proof</u> Recall that we say γ is planar if there is a fixed <u>nonzero</u> vector μ such that $\mu \cdot \gamma(s) \equiv$ constant

$$\tau(s) \equiv 0 \longleftrightarrow \frac{dB}{ds} \equiv 0 \longleftrightarrow B(s) \equiv \text{constant} = \mu$$

say.

($\Rightarrow$) suppose $B(s) \equiv \mu$ a constant.

Then $0 \equiv B(\sigma) \cdot T(\sigma) = B(s) \cdot T(\sigma) = \mu \cdot T(\sigma)$

Then $\mu \cdot \gamma(s) = \mu \cdot \gamma(0) + \int_0^s \mu \cdot T(\sigma)\, d\sigma$

$= \mu \cdot \gamma(0) =$ Constant. (PLANAR)

($\Leftarrow$) Suppose $\mu \cdot \gamma(s) \equiv$ constant, $\mu \neq 0$.

$\Rightarrow$ $\mu \cdot \gamma'(s) = \mu \cdot T(s) \equiv 0$

$\Rightarrow$ $\mu \cdot T'(s) = \kappa(s)\, \mu \cdot N(s) \equiv 0$

Since $\kappa(s) \neq 0$ (nondegeneracy),

$$\mu \cdot N(s) \equiv 0.$$

$$\Rightarrow 0 \equiv \mu \cdot N'(s) = -\kappa(s)\mu \cdot T(s) + \tau(s)\,\mu \cdot B(s)$$

$$= 0 + \tau(s) \cdot (\mu \cdot B(s))$$

Since $\mu \cdot T(s) \equiv 0$ and $\mu \cdot N(s) \equiv 0$, it is necessary that $\mu \cdot B(s) \neq 0$ for <u>any</u> s. Otherwise the constant vector

$$\mu = (\mu \cdot T(s))\,T(s) + (\mu \cdot N(s))\,N(s) + (\mu \cdot B(s))\,B(s)$$

$$= 0$$

Hence $\tau(s) \equiv 0$. $\blacksquare$

Kinematics of particles in $\mathbb{R}^3$.

Suppose a particle in $\mathbb{R}^3$ traces a trajectory $\gamma(t)$ where t = time. Let $s(t)$ = arc length along trajectory traversed in time t

$$= \int_0^t \left\| \frac{d\gamma}{dt} \right\| \cdot dt$$

$\frac{ds}{dt}$ = speed, denoted by v

Then $\underline{V}(t)$ = velocity

$$= \frac{d\gamma}{dt}$$

$$= \frac{d\gamma}{ds}\frac{ds}{dt}$$

$$= T(s)\frac{ds}{dt}$$

$$= v(s)\,T(s)$$

Let $g(s) = \left[\begin{array}{c|c} F(s) & \gamma(s) \\ \hline 0 & 1 \end{array}\right] \in SE(3)$

Then $\frac{dg}{ds} = g \cdot \left(\begin{array}{c|c} \Omega(s) & e_1 \\ \hline 0 & 0 \end{array}\right) \qquad (*)$

where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$(*)$ is a <u>control system</u> on a Lie group,

controlled by curvature and torsion.
It is very interesting to consider optimal control problems of the form:

$$\text{Min} \int_0^L \left(\kappa^2(s) + \tau^2(s) \right) ds$$

subject to $\kappa(s) > 0$, $s \in [0, L]$

$g(0) = \mathbb{1}_{4\times 4}$ $\quad g(L) = g_1$ prescribed

and

$$\frac{dg}{ds} = g \left[\begin{array}{c|c} \Omega & e_1 \\ \hline 0 & 0 \end{array} \right]$$

We can express everything in the original non-unit speed parametrization t.

$$\frac{dg}{dt} = g \left[\begin{array}{c|c} v\Omega & v e_1 \\ \hline 0 & 0 \end{array} \right]$$

where v = speed (as a function of t)