Lecture 3(a) (Contraction Mapping, Existence Uniqueness)

In this lecture we discuss the central idea of a contraction mapping and associated fixed point theorem due to Stefan Banach. This is the tool used to prove the existence-uniqueness theorem for ordinary differential equations.

Definition: Let (S, d) be a metric space and let $f : S \to S$ be a map. We say f is a contraction if there exists $\rho \in (0,1)$ such that,

$$d(f(x), f(y)) \leq \rho\, d(x,y) \quad \forall\, x, y \in S$$

Example: Let $S = \mathbb{R}^n$, and let $\|\cdot\|_\infty$ be defined by $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \quad x \in \mathbb{R}^n$.

Let $d(x,y) = \|x - y\|_\infty$

Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map (matrix) satisfying $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \forall\, i \in \{1, 2, \dots, n\}$

Diagonal dominance

~~Then A is a contraction.~~

Then $A = D^{-1}(L+U)$ is a contraction CORRECTION

Definition: We say $x^* \in S$ is a fixed-point of a mapping $f : S \to S$ provided $f(x^*) = x^*$.

Remark: The notion of a fixed point is important in economics (game theory), and many other fields.

Definition A sequence $\{x_k : k=1,2,\cdots\} \subset S$ a metric space with metric d, is said to be convergent, if $\exists\ x^* \in S$ such that $\lim\limits_{n\to\infty} d(x_n, x^*) = 0$. In that case x^* is unique (~~proof~~ $\rightarrow$ use triangle inequality for the metric), and hence we can write $x^* = \lim\limits_{n\to\infty} x_n$.

Definition A sequence $\{x_k : k=1,2,\cdots\} \subset S$ a metric space with metric d, is said to be a <u>Cauchy sequence</u>, if

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0$$

~~Defn~~ <u>Show that</u> : Every <u>Convergent sequence</u> is a <u>Cauchy sequence</u>. The converse is <u>not true</u> in general.

<u>Definition</u> A metric space (S, d) is said to be <u>complete</u> if every Cauchy sequence in S is convergent in S.

<u>Example</u> $S = \mathbb{R}$ with $d(x,y) = |x-y|$ is a complete metric space.

Because of this example, $\mathbb{R}^n$ is also a complete metric space if we consider $d(x,y) = \sum\limits_{i=1}^{n} |x_i - y_i|$.

Norbert Wiener and (later) Stefan Banach focused attention on infinite dimensional vector spaces of functions that have a norm such that the associated metric is complete. Initially these spaces came to be known as Wiener-Banach spaces, and now simply Banach spaces. Given any norm $\|\cdot\|$ on a vector space V, associate a metric,

$$d(x, y) = \|x - y\| \quad , \quad x, y \in V.$$

Theorem [Banach] Let X be a Banach space and let $S \subset X$ be a <u>closed</u> subset. Let $f: S \to S$ be a mapping such that, for some $\rho \in (0,1)$,

$$\|f(x) - f(y)\| \leq \rho \|x - y\| \quad \forall\, x, y \in S.$$

(i.e. f is a contraction in the metric $d(x,y) = \|x-y\|$).

Then $\exists$ a unique $x^* \in S$ s.t. $f(x^*) = x^*$ (fixed point). Further, this fixed point can be obtained the method of successive approximations (<u>Banach iteration</u>) ☒

Before we proceed to ~~the proof~~ of Banach's theorem we need a few basics.

Ball : An open ball in a metric space (S, d) centered at $x_0 \in S$ and of radius $\varepsilon > 0$ is denoted

$$B_\varepsilon(x_0) = \{x \in S : d(x, x_0) < \varepsilon\}$$

We say a set $P \subset S$ is open (in the given metric) if given $x \in P$, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset P$.

A set $T \subset S$ is closed if

$$P = T^c = \{y \in S : y \notin P\}$$

is open.

A closed set has the property that for every convergent sequence $\{x_k : k = 1, 2, \cdots\}$ contained in the set with limit x^*, the limit x^* is also in the same set.

Proof of Banach's theorem:

Let $x_1 \in S$. Define the sequence $\{x_k : k \geq 1\}$ by

$$x_{k+1} = f(x_k).$$

By hypothesis, $\{x_k\} \subset S$.

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Lecture 4 (part i)

In this lecture, we discuss the existence and uniqueness of solutions to ordinary differential equations. The central idea is the Contraction Mapping - Fixed Point Theorem due to S. Banach.

Theorem 1: Let X be a Banach space and let $S \subset X$ be a closed **subset**. Let $f: S \to S$ be a mapping such that, for some $0 \leq \rho < 1$,

$$\| f(x) - f(y) \| \leq \rho \, \| x - y \| \quad \forall \; x, y \in S.$$

(such a map is called **a contraction**.)

Then there is a unique $x^* \in S$ such that $x^* = f(x^*)$. Further this **fixed point** can be obtained by the method of successive approximations (Banach iterations).

Proof: Let $x_1 \in S$. Define the sequence $\{ x_k : k \geq 1 \}$ by

$$x_{k+1} = f(x_k).$$

By hypothesis, $\{x_k\} \subset S$.

note: Stefan Banach was a central figure in the mathematical life of Poland in pre-WWII era. see: http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Banach.html

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$$\| x_{k+1} - x_k \| = \| f(x_k) - f(x_{k-1}) \|$$

$$\leq \rho \, \| x_k - x_{k-1} \|$$

$$\leq \rho^2 \, \| x_{k-1} - x_{k-2} \|$$

(repeating the previous step)

$$\vdots$$

$$\leq \rho^{k-1} \, \| x_2 - x_1 \|$$

Hence, $\| x_{k+r} - x_k \| = \| x_{k+r} - x_{k+r-1} + x_{k+r-1} \cdots - x_k \|$

$$\leq \| x_{k+r} - x_{k+r-1} \| + \| x_{k+r-1} - x_{k+r-2} \| + \cdots + \| x_{k+1} - x_k \|$$

$$\leq (\rho^{k+r-2} + \rho^{k+r-3} + \cdots + \rho^{k-1}) \, \| x_2 - x_1 \|$$

for $k \geq 1$

$$\leq \rho^{k-1} \sum_{j=0}^{\infty} \rho^j \cdot \| x_2 - x_1 \|$$

$$= \frac{\rho^{k-1}}{1-\rho} \, \| x_2 - x_1 \|$$

$\longrightarrow 0$ as $k \to \infty$, since $\rho < 1$

Hence $\{x_k\}$ is a Cauchy sequence. Since X is a Banach space there is x^* such that $x_k \to x^*$. But S is closed. Therefore $x^* \in S$. To see that x^* is a fixed point,

$$\|x^* - f(x^*)\| \le \|x^* - x_k\| + \|x_k - f(x^*)\|$$
$$\le \|x^* - x_k\| + \rho\,\|x_{k-1} - x^*\|$$
$$\to 0 \quad \text{as } k \to \infty.$$

Hence $\|x^* - f(x^*)\| = 0 \Rightarrow x^* = f(x^*)$.

To prove uniqueness suppose $y^* \in S$ is another fixed point.

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\|$$
$$\le \rho\,\|x^* - y^*\|$$

But $\rho < 1$. So $\|x^* - y^*\| = 0 \Rightarrow x^* = y^*$. ■

If the mapping f were to depend on a parameter in a continuous way, so does the fixed point.

<u>Theorem 2</u> [Continuity of Fixed Point w.r.t. Parameter].

Let $\circledR{H}$ be a metric space with metric d. Let X be a Banach space and let $S \subset X$ be a closed subset, such that

$$f : \circledR{H} \times S \to S$$

has the following properties :

(i) Each <u>partial map</u>

$$f_\theta : S \to S \qquad \theta \in \Theta$$

(defined by $f_\theta(x) = f(\theta, x)$)
is a <u>contraction</u> with parameter $\rho < 1$, independent of θ.

(ii) For each $x \in S$, the <u>partial map</u>

$$f^x : \Theta \to S \qquad x \in S$$

(defined by $f^x(\theta) = f(\theta, x)$),
is continuous, i.e. given $\epsilon > 0$ there exists $\delta_x > 0$ such that

$$d(\theta, \theta') < \delta_x \Rightarrow \|f^x(\theta) - f^x(\theta')\| < \epsilon.$$

Then, the map $\theta \mapsto x^*_\theta$ which assigns to each $\theta \in \Theta$, the (unique) fixed point x^*_θ of f_θ, is continuous.

<u>Proof</u>: $\|x^*_\theta - x^*_{\theta'}\| = \|f_\theta(x^*_\theta) - f_{\theta'}(x^*_{\theta'})\|$

$$\leq \|f_\theta(x^*_\theta) - f_\theta(x^*_{\theta'})\| + \|f_\theta(x^*_{\theta'}) - f_{\theta'}(x^*_{\theta'})\|$$

$$\leq \rho \|x^*_\theta - x^*_{\theta'}\| + \|f^{x^*_{\theta'}}(\theta) - f^{x^*_{\theta'}}(\theta')\|.$$

Hence $\|x_{\theta^*} - x^*_{\theta'}\| \leq \frac{1}{1-\rho} \|f^{x^*_{\theta'}}(\theta) - f^{x^*_{\theta'}}(\theta')\|$

$$< \frac{\varepsilon}{1-\rho} \text{ whenever } d(\theta', \theta) < \delta_{x^*_{\theta'}}$$

This proves continuity of the fixed point.

<u>Example</u> (Jacobi's algorithm)

The linear equation in $\mathbb{R}^n$,

$$Ax = b$$

where A is a square matrix can be identified as the fixed-point **problem**

$$x = -D^{-1}(L+U)x + D^{-1}b$$

where, $A = L + D + U$ denotes the decomposition into strictly lower triangular, diagonal, and strictly upper triangular parts and we <u>assume</u> D is invertible.

Jacobi's algorithm, to solve this problem:

$$x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b$$

is a special case of the Banach iteration and, to guarantee convergence, it is sufficient that A be <u>diagonally dominant</u>:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

Then we can take $\rho = \max_i \left(\dfrac{1}{|a_{ii}|} \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right) \right)$,

making $f(x) = -D^{-1}(L+U)x + D^{-1}b$, a contraction on all of $\mathbb{R}^n$.

Consider the scalar equation

Example
$$g(x) = x^2 - b = 0 \qquad b > 0.$$

Let $y = 1 - x$. The problem of finding the (positive) square root of b is a fixed-point problem
$$y = \tfrac{1}{2}\left[(1-b) + y^2\right] = f(y).$$

Suppose $|1-b| < \rho < 1$.

Then f maps the **closed** subset $S = \{ y : |y| \le \rho \} \subset \mathbb{R}$ into itself and it is a contraction on S with parameter ρ. Thus the algorithm
$$y_{n+1} = \tfrac{1}{2}\left[(1-b) + y_n^2\right]$$

converges for $|1-b| < \rho < 1$.

It is equivalent to

$$x_{n+1} = x_n - \tfrac{1}{2} x_n^2 + \tfrac{1}{2} b.$$

Exercise

How does it <u>compare</u> with Newton's method?

We are interested in (and ready for) applying Banach's theorem to o.d.e's.

Let $\dot{x} = f(t, x)$ be a **non-autonomous** ordinary differential equation. A continuously differentiable function $x(t)$ is a solution if,

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma}$$

$t \in [t_0, t_0+\delta]$, for some $\delta > 0$. We aim to show existence and uniqueness of solutions to the above integral equation in a suitable function space, the space $(X, \|\cdot\|_X)$ below.

For any $\delta > 0$, the space

$$X = \{\psi : [t_0, t_0+\delta] \to \mathbb{R}^n \mid \psi \text{ continuous}\}$$

with norm

$$\|\psi\|_X = \max_{t \in [t_0, t_0+\delta]} \|\psi(t)\|$$

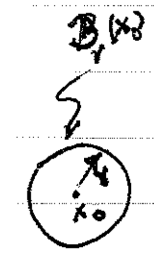
where $\|\cdot\|$ in $\mathbb{R}^n$ is <u>any</u> norm, is a Complete normed linear space, i.e. Banach space. (<u>Proof of completeness</u> $\rightarrow$ <u>exercise</u>)

$\longrightarrow$ <u>See appendix B</u>, Example B.1 (Khalil 3rd ed.)

<u>Theorem 3</u> (Local Existence and Uniqueness)

Consider the system

$$\dot{x}(t) = f(t, x),$$



Let f be piecewise continuous in t and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\, \|x-y\|$$

$\forall\ x, y \in B_r(x_0) = \{x : \|x - x_0\| \leq r\}$ and $\forall\ t \in [t_0, t_1]$. ~~Then~~ there is some $\delta > 0$ such that the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma \qquad t \in [t_0, t_0+\delta]$$

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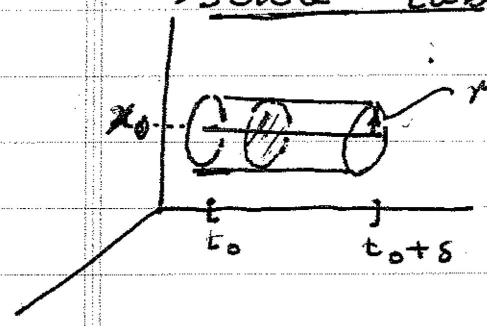
has a unique solution x in X. It is differentiable with respect to t and $\dot{x}(t)$ agrees with $f(t, x(t))$ at all points of continuity in t of f.

Proof: Define $P : X \to X$

$$(Px)(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma \qquad t \in [t_0, t_0+\delta]$$

Let $x_0(\cdot)$ denote the constant function belonging to X, $x_0(t) \equiv x_0$ for $t \in [t_0, t_0+\delta]$.

Let $S := \{ x \in X \mid \|x - x_0\|_X \le r \}$ the solid tube in the figure. It is a closed ball in X.



We will be choosing $\delta > 0$ such that $t_0 + \delta \le t_1$

Observation (i) Since f is piecewise continuous in t, so is $\|f(t,x)\|$ for every x. Thus $\|f(t, x_0)\|$ is bounded on $[t_0, t_1]$. We set

$$h = \max_{t \in [t_0, t_1]} \| f(t, x_0) \| .$$

Observation (ii) Let $x(\cdot) \in S$. Then for $t \le t_0+\delta$

$$\|(Px)(t) - x_0\| = \left\| \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma \right\|$$

(we use triangle inequality here)

$$\le \int_{t_0}^{t} \| f(\sigma, x(\sigma)) \| \, d\sigma$$

$$\le \int_{t_0}^{t} \| f(\sigma, x(\sigma)) - f(\sigma, x_0) \| \, d\sigma$$

$$+ \int_{t_0}^{t} \| f(\sigma, x_0) \| \, d\sigma$$

(triangle inequality)

$$\le \int_{t_0}^{t} \left(L \, \| x(\sigma) - x_0 \| + h \right) d\sigma$$

(by Lipschitz condition & obs (i))

$$\le \int_{t_0}^{t} (L \cdot r + h) \, d\sigma \qquad (\text{since } x(\sigma) \in S)$$

$$= (t - t_0)(Lr + h)$$

$$\le \delta \cdot (Lr + h).$$

Hence $\| Px - x_0 \|_X = \max\limits_{t_0 \le t \le t_0 + \delta} \| (Px)(t) - x_0 \|$

$$\le \delta \cdot (Lr + h)$$

$$\le r, \quad \text{if } \delta \le \frac{r}{Lr + h}.$$

So choosing $\delta \le \dfrac{r}{Lr+h}$ ensures that <u>P maps S into S.</u>

In this case,

<u>Observation</u> (iii) P is a contraction on S.

To see this, let $x, y \in S$.

$$\| (Px)(t) - (Py)(t) \| = \left\| \int_{t_0}^{t} [f(\sigma, x(\sigma)) - f(\sigma, y(\sigma))] \, d\sigma \right\|$$

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$$\leq \int_{t_0}^{t} \| f(\sigma, x(\sigma)) - f(\sigma, y(\sigma)) \| d\sigma$$

$$\leq \int_{t_0}^{t} L \cdot \| x(\sigma) - y(\sigma) \| d\sigma \qquad \text{(by Lipschitz condition)}$$

$$\leq L \cdot (t - t_0) \; \| x(\cdot) - y(\cdot) \|_{X}$$

Hence $\|Px - Py\|_{X} \leq L \cdot \delta \; \|x - y\|_{X}$

$$\leq \rho \; \|x - y\|_{X},$$

if $\delta \leq \frac{\rho}{L}$

Thus choosing $\rho < 1$, and

$$\delta \leq \min \left(t_1 - t_0, \; \frac{r}{Lr + h}, \; \frac{\rho}{L} \right)$$

ensures that $P: S \to S$ is a contraction mapping.

Hence, by the theorem of Banach there is a unique fixed point for P in S, the solution to the integral equation. We can actually show that this is the only solution in X.

Since $x_0 \in B(x_0, r)$, any (continuous) solution $x(t)$ must ~~leave~~ lie inside $B(x_0, r)$

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for some non trivial interval of time. Suppose $x(t)$ leaves $B(x_0, r)$ and $t_0 + \mu$ is the first instant of time that $x(t)$ intersects $\partial B(x_0, r)$ the boundary of $B(x_0, r)$. Then

$$\| x(t_0+\mu) - x_0 \| = r.$$

On the other hand, $\forall\ t \le t_0 + \mu$,

$$\| x(t) - x_0 \| \le \int_{t_0}^{t} (Lr + h)\, ds \qquad \text{(see obs(ii))}$$

so that
$$r = \| x(t_0+\mu) - x_0 \|$$

$$\le (Lr+h)\mu \Rightarrow \mu \ge \frac{r}{Lr+h} \ge \delta.$$

Hence the solution starting at x_0 stays in $B(x_0, r)$ and hence in S during $[t_0, t_0+\delta]$. Consequently uniqueness of solution in S $\Rightarrow$ uniqueness in X. ▨

$\boxed{\text{Here Banach iteration} = \text{Picard–Lindelöf iteration}}$

Notice that the map P in the local existence and uniqueness theorem depends on x_0 in a continuous way.

Corollary 4 Let Θ be a metric space that parametrizes a family of differential equations and initial conditions. Suppose the parametrization is such that the conditions of Theorem 2 are satisfied. Then by Theorem 2 the solutions obtained in Theorem 3 depends continuously on x_0 and more generally $\theta \in \textcircled{H}$. ∎

Corollary 4 is a very useful result to keep in mind. The following lemma leads to comparison of solutions

Gronwall-Bellman Inequality / Lemma 5

<How to "solve" an inequality?>

Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies

(implicit)
$$y(t) \le \lambda(t) + \int_a^t \mu(s) y(s) ds, \qquad a \le t \le b,$$

then

<explicit>
$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau)d\tau\right] ds, \qquad a \le t \le b.$$

In particular if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \le \lambda \exp\left[\int_a^t \mu(\tau)d\tau\right].$$

If, in addition, $\mu(t) \equiv \mu \ge 0$ is a constant, then

$$y(t) \le \lambda \exp[\mu(t-a)]$$

Proof: Let $z(t) = \int_a^t \mu(s) y(s) ds$ and

$$v(t) = z(t) + \lambda(t) - y(t) \ge 0.$$

Then, z is differentiable and

$$\dot{z}(t) = \mu(t) y(t)$$
$$= \mu(t) z(t) + \mu(t)\lambda(t) - \mu(t) v(t).$$

This scalar equation has the solution, (since $z(a) = 0$)

$$z(t) = \int_a^t \phi(t,s) [\mu(s)\lambda(s) - \mu(s)v(s)] ds$$

where $\phi(t,s) = \exp\left[\int_s^t \mu(\tau)\,d\tau\right] > 0$

By hypothesis, $\int_a^t \phi(t,s)\,\mu(s)\,v(s)\,ds \geq 0$. Therefore,

$$z(t) \leq \int_a^t \exp\left(\int_s^t \mu(\tau)\,d\tau\right) \cdot \lambda(s)\mu(s)\,ds$$

and, since $y(t) \leq \lambda(t) + z(t)$, the proof of the general case is completed.

The remaining cases amount to computing integrals — left to the reader. ■

Corollary 6 $f(t,x)$ is piecewise continuous in t and Lipschitz in x on $[t_0\ t_1] \times W$ with Lipschitz constant L, where $W \subset \mathbb{R}^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y); \quad y(t_0) = y_0$$

and

$$\dot{z} = f(t,z) + g(t,z); \quad z(t_0) = z_0$$

such that $y(t), z(t) \in W$, $\forall\ t \in [t_0\ t_1]$.
Suppose the perturbation is bounded:

$$\|g(t,x)\| \leq \mu \quad \forall\ (t,x) \in [t_0\ t_1] \times W$$

for some $\mu \geq 0$, and

$$\|y_0 - z_0\| \leq \gamma.$$

Then,

$$\|y(t) - z(t)\| \leq \gamma \exp[L(t-t_0)] + \frac{\mu}{L}\left\{e^{L(t-t_0)} - 1\right\}$$
$$\forall\ t \in [t_0, t_1]$$

<u>Proof</u>: $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$

$$z(t) = z_0 + \int_{t_0}^{t} \left[f(s, z(s)) + g(s, z(s)) \right] ds.$$

Then,

$$\| y(t) - z(t) \| \le \| y_0 - z_0 \| + \int_{t_0}^{t} \| f(s, y(s)) - f(s, z(s)) \| ds$$

$$+ \int_{t_0}^{t} \| g(s, z(s)) \| ds$$

$$\le \gamma + \mu \cdot (t - t_0) + \int_{t_0}^{t} L \, \| y(s) - z(s) \| ds.$$

By the Gronwall–Bellman inequality,

$$\| y(t) - z(t) \| \le \gamma + \mu (t - t_0) + \int_{t_0}^{t} L \cdot (\gamma + \mu (s - t_0)) \cdot e^{L \cdot (t-s)} ds$$

$$= \gamma + \mu \cdot (t - t_0) - \gamma - \mu \cdot (t - t_0) + \gamma \cdot e^{L \cdot (t - t_0)} + \int_{t_0}^{t} \mu \cdot e^{L \cdot (t - s)} ds$$

; (integration by parts)

$$= \gamma e^{L \cdot (t - t_0)} + \frac{\mu}{L} \left(e^{L \cdot (t - t_0)} - 1 \right) \qquad \blacksquare$$

<u>Remark</u> In these applications as in the original Gronwall–Bellman inequality one is turning an implicit inequality explicit — in effect "solving the inequality".

Remark

Corollary 6 allows us to quantitatively estimate the effects of perturbations — in initial conditions and the dynamics. Such estimates are useful to keep in mind — all models of physical systems display errors due to various unavoidable approximations.

Theorem 7 (Global Existence and Uniqueness)

Suppose $f(t,x)$ in Theorem 3 is piecewise continuous in t, and satisfies

$$\|f(t,x_0)\| \leq h,$$

global Lipschitz $\rightarrow$ $\|f(t,x)-f(t,y)\| \leq L \cdot \|x-y\|$

$\forall\, x, y \in \mathbb{R}^n, \; t \in [t_0, t_1]$

then the o.d.e.

$$\dot{x}(t) = f(t,x) \quad \text{with } x(t_0)=x_0$$

has a unique solution on $[t_0, t_1]$.

Proof: We show how to modify the proof of Theorem 3. There we now let r be arbitrarily large (due to the global Lipschitz condition) so that $\dfrac{r}{Lr+h} > \dfrac{\rho}{L}$.

(by taking $r > \frac{\rho}{(1-\rho)}\frac{h}{L}$)

Thus we only need $\delta \leq \min\left\{t_1-t_0, \frac{\rho}{L}\right\}$ for $\rho < 1$. If $t_1-t_0 \leq \frac{\rho}{L}$, we could let $\delta = t_1 - t_0$ and we are done.

If not, choose $\delta = \frac{\ell}{L}$, divide $[t_0, t_1]$ into a finite number of sub-intervals of length $\delta = \frac{\ell}{L}$ and repeat that many times the arguments of Theorem 3. This completes the proof ▨

EXAMPLE

$$\dot{x} = -x^3$$

does not satisfy a global Lipschitz condition. But there is a unique solution

$$x(t) = \operatorname{sgn}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2 \cdot (t - t_0)}} \qquad t \geq t_0$$

through $x(t_0) = x_0$.

The essential idea here is that if $x(0) = a$, the set $\{x : |x| \leq a\}$ is a positively invariant, closed and bounded ~~solution~~ set for the dynamics $\dot{x} = -x^3$. This idea can be generalized to

Theorem

$f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz on a domain $D \subset \mathbb{R}^n$. Suppose there is a closed and bounded set $W \subset D$ such that $x(0) = x_0 \in D$ and $f|_{\partial W}$ points into W. Then there is a unique solution $x(t)$ of $\dot{x} = f(x)$ such that $x(0) = x_0$

Proof : EXERCISE.

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Some Definitions and Properties Pertaining to Lipschitz Condition.

open connected

Def^n (a) f is $\boxed{\text{locally}}$ Lipschitz on a domain $D \subset \mathbb{R}^n$ if each point p of D has neighborhood (i.e ball $B_\varepsilon(p)$ surrounding p, $\varepsilon > 0$) such that

$$\| f(x) - f(y) \| \leq L_{B_\varepsilon} \cdot \| x - y \|$$

$\forall\, x, y \in B_\varepsilon$ and $L_{B_\varepsilon} > 0$.

Here $B_\varepsilon(p)$ is open ball

Def^n (b) f is Lipschitz on a set W if

$$\| f(x) - f(y) \| \leq L \cdot \| x - y \|, \quad \forall\, x, y \in W.$$

Properties

(a) f is locally Lipschitz on a domain D $\Rightarrow$ f is continuous on D.

(b) f is Lipschitz on domain $D \Rightarrow f$ is uniformly continuous on D

(c) Converse of (a) is not true:
Consider $f(x) = x^{1/3}$ on $(-1, 1)$

(d) f is locally Lipschitz on domain D $\not\Rightarrow$ f is Lipschitz on D (due to lack of uniformity of the Lipschitz constant

(e) f is locally Lipschitz on domain D $\Rightarrow$ Lipschitz on every closed ~~and~~ bounded ~~subset~~ of D.

(f) f is continuously differentiable $\Rightarrow$ f is locally Lipschitz. Converse ~~is~~ far from true.