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Lecture 10

Nonlinear Systems and Feedback

Consider the system

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Suppose $f(0,0) = 0$ and f is C^1. Let $A = \left(\frac{\partial f}{\partial x}\right)_{|(0,0)}$ and let $B = \left(\frac{\partial f}{\partial u}\right)_{|(0,0)}$.

Hypothesis 1

Let K be such that

$$\text{spectrum}\,(A + BK) \subseteq \mathbb{C}^-$$

Consider the closed loop system

$$\dot{x} = f(x, u)$$

$$u = Kx.$$

Thus $\dot{x} = \tilde{f}(x) = f(x, Kx)$.

Clearly $\tilde{f}(0) = 0$. The linearization of the closed loop system at the origin is

$$\dot{z} = \tilde{A} z,$$

where,

$$\tilde{A} = \left(\frac{\partial \tilde{f}}{\partial x}\right)_0$$

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But $\left(\frac{\partial \tilde{f}}{\partial x}\right)\Big|_{x=0} = \frac{\partial f}{\partial x}(x, Kx)\Big|_{x=0}$

$$= D_1 f\Big|_{x=0} + D_2 f \cdot K\Big|_{x=0} \qquad \text{(chain rule)}$$

$$= (A + BK).$$

By hypothesis 1, and the indirect method of Lyapunov, the origin is an asymptotically stable equilibrium of the closed loop system.

Remark : A sufficient condition for hypothesis 1 to hold is that the pair $[A, B]$ is controllable < Recall : the eigenvalue/pole placement theorem>

We see that $u = Kx$ a linear feedback law, can be stabilizing. The region of attraction may be estimated by

(i) solving $(A+BK)^T P + P(A+BK) = -Q$ for a $Q = Q^T > 0$

(ii) let $g(x) = f(x, Kx) - (A+BK)x$

and observe that $\|g(x)\|_2 \leq \gamma \|x\|_2$ for $\|x\|_2 < r$ and γ can be made arbitrarily small by choosing r small enough

Then $B_r(0)$ is an estimate of the region of attraction provided $(-\lambda_{min}(Q) + 2\gamma \|P\|_2) < 0$

< This is the argument used in pages 4 & 5 of Lecture Notes 6 (part iii) >.

But $B_r(0)$ may be too small for practical purposes.

One approach to overcome this problem is to use feedback and changes of coordinates in input space and state space to exactly linearize a nonlinear control system in a (sufficiently large) neighborhood of any equilibrium point. Even if the linearization does not have asymptotic stability, controllability can ensure the existence of an additional linear feedback to stabilize the system / equilibrium.

Definition

Let $\dot{y} = f(y) + G(y)u$ where

$f(0) = 0$, $G(y) = [g_1(y), \cdots, g_m(y)]$ and

$g_i(0) = 0$.

We say that this system is exact, state feedback linearizable if there exists

$$T : U \overset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$0 \in U$, T is C^∞ and invertible, T^{-1} exists and is C^∞, and functions $\alpha(\cdot)$ & $\beta(\cdot)$ such that under the change of coordinates by T,

$$x = T(y)$$

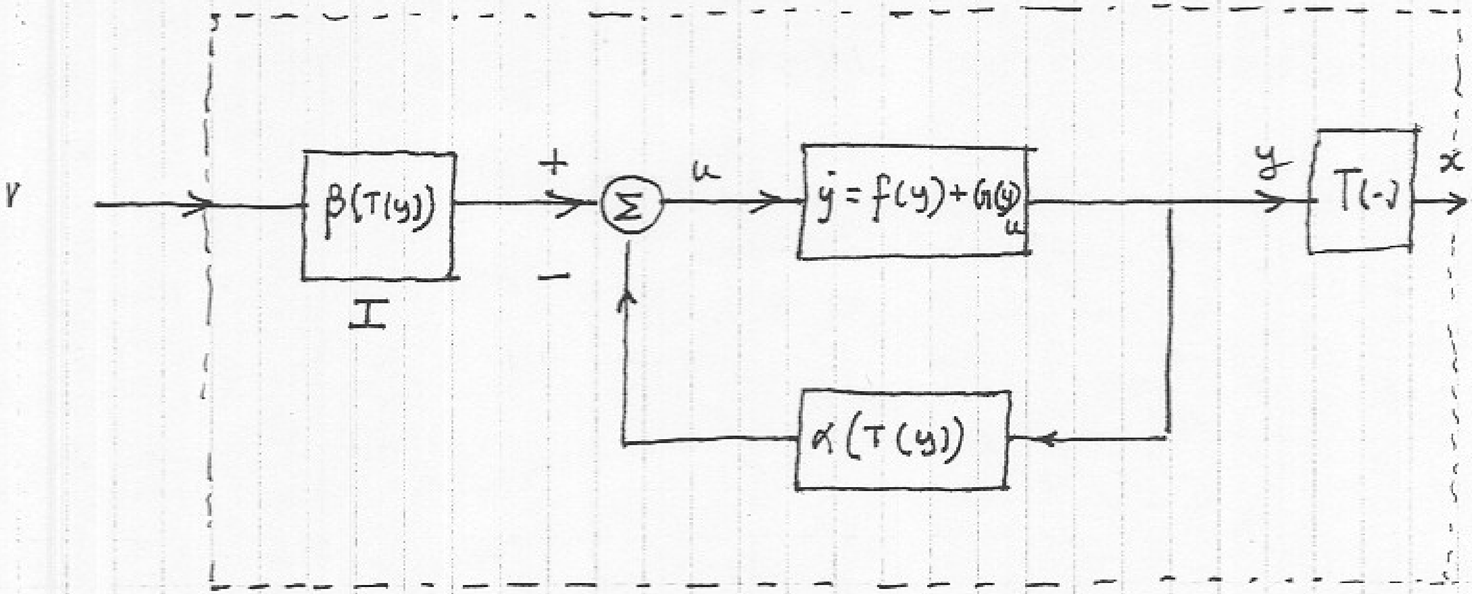
satisfies

$$\begin{aligned} \dot{x} &= Ax + B\beta^{-1}(x)[u - \alpha(x)] \\ &\triangleq Ax + Bv \end{aligned}$$

where $v \triangleq \beta^{-1}(x) \cdot [u - \alpha(x)]$

and $[A, B]$ is controllable

[We then seek $\beta(\cdot)$ such that $\beta(T(y))$ is an invertible $m \times m$ matrix at every y.]



The system outside the dotted line is linear.

By chain rule

$$
\begin{aligned}
\dot{x} &= \frac{\partial T}{\partial y}\,\dot{y} \\
&= \frac{\partial T}{\partial y}\left(f(y) + G(y)\,u\right) \\
&= A x + B \beta^{-1}(x)\,[u - \alpha(x)] \\
&= A\,T(y) + B \beta^{-1}(T(y))\,[u - \alpha(T(y))]
\end{aligned}
$$

$\forall \; y \in U$.

Set $u \equiv 0$

$$
\Rightarrow \quad \left\{
\begin{aligned}
\frac{\partial T}{\partial y} f(y) &= A\,T(y) - B\,[\beta(T(y))]^{-1}\alpha(T(y)) \\
\frac{\partial T}{\partial y} G(y) &= B\,[\beta(T(y))]^{-1}
\end{aligned}
\right.
$$

Consider the single input case $(m=1)$

with canonical form $A = A_c$; $B = B_c$ & $G \hat{=} g$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then the conditions above on T take the form

EQN #		EQN #	
(1a)	$\frac{\partial T_1}{\partial y} \cdot f(y) = T_2(y)$	(1b)	$\frac{\partial T_1}{\partial y} \cdot g(y) = 0$
(2a)	$\frac{\partial T_2}{\partial y} \cdot f(y) = T_3(y)$	(2b)	$\frac{\partial T_2}{\partial y} \cdot g(y) = 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
((n-1)a)	$\frac{\partial T_{n-1}}{\partial y} \cdot f(y) = T_n(y)$	((n-1)b)	$\frac{\partial T_{n-1}}{\partial y} \cdot g(y) = 0$
(n a)	$\frac{\partial T_n}{\partial y} \cdot f(y) = -\frac{\alpha}{\beta}$	(n b)	$\frac{\partial T_n}{\partial y} \cdot g(y) = \frac{1}{\beta}$

(assuming $\beta \neq 0$)

Define the operation (Lie derivative)

$$L_f \cdot h = \frac{\partial h}{\partial x} \cdot f$$

where h is a scalar function and f is a vector field.

Define $L_f^0 h \triangleq h$

and

$$L_f^{k+1} h = L_f \left(L_f^k h \right)$$

With these definitions we have relations,

$$T_k = L_f^{k-1} T_1 \qquad k = 2, 3, \ldots, n$$

And we have the equations

$$\boxed{L_g L_f^k T_1 = 0} \qquad k = 0, 1, 2, \ldots, (n-2)$$

If we can solve these equations for T_1, then by using the recursions ~~..~~ above we can define T_k, $k = 2, \ldots, n$ and

$$\beta = \left(L_g T_n \right)^{-1}$$

If β ~~is invertible~~ $\neq 0$ then $\alpha = - \dfrac{L_f T_n}{L_g T_n}$

what about solvability of $\underbrace{\text{equations for } T_1}$?

Define

$$ad_f\, g = \left(\frac{\partial g}{\partial x}\right) f - \left(\frac{\partial f}{\partial x}\right) g$$

and also

$$ad_f^0\, g \triangleq g$$

$$ad_f^{k+1}\, g \triangleq ad_f\,(ad_f^k\, g).$$

<u>Theorem</u> ~~Then~~ There exists, (locally in a suitable neighborhood of 0) a function T_1 s.t.

$$L_g L_f^k T_1 = 0 \qquad k = 0, 1, 2, \ldots, (n-2)$$

<u>iff</u>

(i) $\{g, ad_f\, g, ad_f^2\, g, \ldots, ad_f^{n-1}\, g\}$ is a set of linearly independent vector fields

(ii) $\{g, ad_f\, g, ad_f^2\, g, \ldots, ad_f^{n-2}\, g\}$ is a ~~~~ ~~~~ set of vector fields satisfying the involutive property

$p(x),\ q(x) \in$ this set

$$\Rightarrow \left(\frac{\partial q}{\partial x}\right) p(x) - \left(\frac{\partial p}{\partial x}\right) q(x) \text{ also belongs}$$

to this set $\blacksquare$

<u>Remark</u> : This existence result is a consequence of Frobenius' Theorem in differential geometry.

Example:

$$\dot{y} = f(y) + g(y)u$$

$$\cancel{f(y) = \begin{bmatrix} y_2 \\ -a[(1+y_3) \end{bmatrix}}$$

$$f(y) = \begin{bmatrix} y_2 \\ -a\sin(y_1) - b(y_1 - y_3) \\ y_4 \\ c(y_1 - y_3) \end{bmatrix} \qquad g(y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix}$$

$d \neq 0$, $d > 0$.

$a, b, c > 0.$

$$L_g T_1 = 0 \quad \Leftrightarrow \quad \frac{\partial T_1}{\partial y_4} = 0$$

$$\Rightarrow \quad \underline{T_1 \text{ independent of } y_4.}$$

$$T_2 = L_f T_1$$

$$= \frac{\partial T_1}{\partial y_1} \cdot y_2 + \frac{\partial T_1}{\partial y_2}\left(-a \sin y_1 - b(y_1 - y_3)\right)$$

$$+ \frac{\partial T_1}{\partial y_3} y_4$$

$$L_g T_2 = 0 \quad \Leftrightarrow \quad \frac{\partial T_1}{\partial y_3} = 0$$

$$\Rightarrow \quad \underline{T_1 \text{ independent of } y_3}$$

$$\Rightarrow \quad T_2 = \frac{\partial T_1}{\partial y_1} \cdot y_2 + \frac{\partial T_1}{\partial y_2}\left(-a \sin y_1 - b(y_1 - y_3)\right)$$

$$T_3 = L_f T_2$$

$$= \frac{\partial T_2}{\partial y_1} y_2 + \frac{\partial T_2}{\partial y_2}\left(-a \sin y_1 - b(y_1 - y_3)\right) + \frac{\partial T_2}{\partial y_3} y_4$$

$$L_g T_3 = 0 \Rightarrow \frac{\partial T_3}{\partial y_4} = 0 \Rightarrow \frac{\partial T_2}{\partial y_3} = 0$$

$$\cancel{\Rightarrow b \frac{\partial T_1}{\partial y_2} = 0 \Rightarrow \frac{\partial T_1}{\partial y_2} = 0 \Rightarrow T_1 \text{ indep of } y_2}$$

$\Rightarrow \quad b \dfrac{\partial T_1}{\partial y_2} = 0 \quad \Rightarrow \quad \dfrac{\partial T_1}{\partial y_2} = 0$

So T_1 is independent of y_2

So $\quad T_1 = T_1(y_1)$

Pick $\quad T_1(y_1) = y_1 \qquad$ (trial).

Then $\quad x_1 = y_1$ //

$$\dot{x}_1 = \dot{y}_1 = y_2 \qquad \text{(from model)}$$

But $\quad \dot{x}_1 = x_2 \qquad$ (linear system)

So $\quad x_2 = T_2(y_2) = y_2$ //

$$x_3 = T_3(y) = \dot{x}_2 = \dot{y}_2$$

$$= -a \sin y_1 - b(y_1 - y_3)$$

(nonlinear model)

$$x_4 = T_4(y) = \dot{x}_3 = -a \dot{y}_1 \cos y_1 - b(\dot{y}_1 - \dot{y}_3)$$

$$= -a y_2 \cos(y_1) - b(y_2 - y_4)$$

Check β & α are well defined //