ENEE 661 SPRING 2017 Homework 2

due date: Tuesday February 14 (beginning of class)

1. Read carefully Lecture 4 Notes, especially pages 30 and 31 and derive equation (4.10). Use this to derive [Exercise 4.1 in Notes],

$$\Phi^{g}_{-\varepsilon}\left(\Phi^{f}_{-\varepsilon}\left(\Phi^{g}_{\varepsilon}\left(\Phi^{f}_{\varepsilon}(x_0)\right)\right)\right) = x_0 + \varepsilon^2 [f, g](x_0) + o(\varepsilon^2)$$

Here $[f, g] = \left(\frac{\partial g}{\partial x}\right) f - \left(\frac{\partial f}{\partial x}\right) g$ is the Jacobi-Lie bracket.

2. Consider the unicycle equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = v \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = u f + v g$$

where f is the STEER vector field and g is the DRIVE vector field. Suppose you have to move the unicycle starting from $(x, y, \theta) = (0, 0, \frac{3\pi}{4})$, to $(1, 0, \frac{\pi}{2})$, a "parking" task.

(i) Use MATLAB to explore the parking algorithm DRIVE, STEER, REVERSE DRIVE, REVERSE STEER, ... to carry out the task.

(ii) Suppose $v \geq 0$ i.e. reverse drive is not available. Can you still carry out the task? What is the difference between the two solutions?

3. Consider the kinematic car (EXAMPLE 4.3) pages 33 and 34 of Lecture 4. Verify the bracket relations. (I also recommend reading the discussion in Edward Nelson's TENSOR ANALYSIS, pages 33-36; link provided on Course Web Page.)
[OPTIONAL - explore the parking algorithm for this setting numerically.]

4. Associated to Hamiltonian systems one encounters infinite dimensional Lie algebras known as Poisson bracket algebras - the Poisson bracket is in fact a Lie bracket.
(i) For $H : \mathbb{R}^{2n} \to \mathbb{R}$ Hamiltonian (function),

$$(q, p) = (q_1, \ldots, q_n, p_1, p_2, \cdots p_n) \mapsto H(q, p),$$

and associated Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q},$$

the Poisson bracket of two (C^∞) functions $\phi : \mathbb{R}^{2n} \to \mathbb{R}$, $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ is given by a function

$$\{\phi, \psi\}(q, p) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial q_i} \frac{\partial \psi}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial \psi}{\partial q_i}$$

Let $\mathcal{F}$ denote the vector space of all C^∞
infinitely differentiable

functions of (q, p). Show that
$$(\mathcal{F}, \{\cdot,\cdot\})$$
is a Lie algebra. (You need to verify that the Lie algebra axioms are satisfied by the definition of Poisson bracket.)

(ii) Let $\mathcal{F}$ denote the vector space of C^∞ functions on $\mathbb{R}^3$. For Hamiltonian function $H : \mathbb{R}^3 \to \mathbb{R}$, one can associate Hamiltonian system (not quite in the sense of (i) above),

$$\dot{m} = m \times \nabla H$$

where
$$\nabla H = \begin{pmatrix} \frac{\partial H}{\partial m_1} \\ \frac{\partial H}{\partial m_2} \\ \frac{\partial H}{\partial m_3} \end{pmatrix} = \text{gradient of } H.$$

Define $\{\phi, \psi\}$ by the formula

cross product

$$\text{for } m \in \mathbb{R}^3, \quad \{\phi, \psi\}(m) = -m \cdot (\nabla\phi \times \nabla\psi)$$

dot product

where $\phi, \psi \in \mathcal{F}$.

Show that $(\mathcal{F}, \{\cdot,\cdot\})$ is a Lie algebra, again infinite dimensional.