ENEE664 Optimal Control   Spring 2004   Homework Set # 4
due back Thursday February 26, 2004

1. Show that $f: \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2^2}{(x_1^2 + x_2^2)} & \text{if } x_1 \neq 0 \\ 0 & \text{if } x_1 = 0 \end{cases}$$

is Gateaux differentiable, but not continuous at $x_1 = x_2 = 0$

2. Show that $f: \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0 \\ \dfrac{2 x_2 e^{-\frac{1}{x_1^2}}}{x_2^2 + \left(e^{-\frac{1}{x_1^2}}\right)^2} & \text{if } x_1 \neq 0 \end{cases}$$

is Gateaux differentiable but not Fréchet differentiable at $(0,0)$.

3. Define $f: \mathbb{R}^2 \to \mathbb{R}^1$ by

$$f(x, y) = \operatorname{sgn}(y) \min(|x|, |y|).$$

Show that, for any $h \in \mathbb{R}^2$,

$$\lim_{t \to 0} \frac{1}{t}\left(f(th) - f(0)\right) = f(h)$$

Is f Fréchet differentiable?

4. Let $X = C[0,1]$ with $\|x\| = \max_{0 \le t \le 1} |x(t)|$.

Let $f: X \to \mathbb{R}$ be $f(x) = (x(1/2))^2$. Find the Fréchet differential of f.

5. Consider the mapping defined on symmetric matrices by $K \mapsto f(K) = A'K + KA - KBB'K + L$.
where A, B, L are constant matrices (arising in a linear-quadratic optimal control problem).
Compute the Fréchet derivative $Df(K_0)$.
When is it invertible?