Here we sketch the basic steps leading up to Taylor's Theorem.

Theorem (Fundamental theorem of integral calculus)

Let X and Y be Banach spaces. Let $U \subset X$ be an open subset. Let $f : U \to Y$ be a C^1 map.

If $x + ty \in U \quad \forall\ t \in [0,1]$

then

$$f(x+y) = f(x) + \int_0^1 Df(x+ty)y\,dt$$

Proof Set $g(t) = f(x+ty) \qquad 0 \le t \le 1$

For $0 < t < 1$, by chain rule,

$$g'(t) = Df(x+ty)\,y$$

Let $h(t) = \int_0^t Df(x+sy)y\,ds \quad + f(x)$

$$0 \le t \le 1.$$

Then, $h'(t) = Df(x+ty)y \qquad 0 < t < 1$

Hence $g'(t) = h'(t) \qquad 0 < t < 1$

$\Rightarrow \quad g(t) = h(t) + \text{constant} \qquad 0 < t < 1$

By continuity wrt t of g & h

$$g(t) = h(t) + \text{constant} \qquad 0 \le t \le 1$$

$g(0) = f(x) = h(0) \Rightarrow$ constant $= 0$.

so $g(1) = f(1)$ ▨

Remarks

(i) In finite dimensions the integral (above) can be taken as Riemann integral.

(ii) In infinite dimensions we proceed as follows

(a) Let Step $([a,b];Y)$ denote the space of all functions on $[a,b]$ with values in Y that take only finitely many distinct values. They form a vector space with sup norm,

$$\|f\| = \sup_{a \le x \le b} \|f(x)\|.$$

(b) Let Regulated $([a,b];Y)$ denote the space of all functions on $[a,b]$ with values in Y obtained by taking the completion of Step $([a,b];Y)$ under the sup norm.

(a continuous function is regulated)

(c) For a function $f \in$ Step $([a,b]:Y)$ ~~…~~ defined by

$$\begin{aligned} f(x) &= c_1 \quad a_0 = a \le x < a_1 \\ &= c_2 \quad a_1 \le x < a_2 \\ &\ \vdots \\ &= c_n \quad a_{n-1} \le x < b = a_n \end{aligned}$$

let

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} (a_{i+1} - a_i)\, c_{i+1}$$

(d) For regulated function $f \in \text{Regulated}([a,b];Y)$

define
$$\int_a^b f(x)\,dx \triangleq \lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right)$$

where $\{f_n\} \subset \text{Step}([a,b];Y)$ such that

$\|f_n - f\| \to 0$ as $n \to \infty$.

(e) This definition does not depend on the approximating sequence $\{f_n\}$ ▨

The fundamental theorem of integral calculus can be extended to 'order n' by repeated application.

Lemma 1

Let $f : [a,b] \to X$ be a C^{n+1} function with values in a Banach space X. Then:

$$D\left[f(t) + (1-t)Df(t) + \cdots + \frac{1}{n!}(1-t)^n D^n f(t)\right]$$
$$= \frac{1}{n!}(1-t)^n D^{n+1} f(t)$$ ▨

Lemma 2 Let $f : [a,b] \to X$ be a C^{n+1} function on an open set $(a,b) \supset [0,1]$ with values in a Banach space X. Then:

$$f(1) - f(0) - f'(0) - \frac{1}{2} f''(0) \cdots - \frac{1}{n!} f^{(n)}(0)$$

$$= \int_0^1 \frac{(1-t)^n}{n!} f^{n+1}(t)\, dt.$$

continuous vector$^{(k)}$ of order k ... [illegible]

Lemma 3

Let u be an $(n+1)$ times differentiable function of a real variable t, defined on a open set containing $[0,1]$ with values in a Banach space X. And such that $\exists$ constant C majorizing $\| D^{n+1} u(t) \|$ for $0 \le t \le 1$. Then:

$$\| u(1) - u(0) - u'(0) - \cdots - \frac{1}{n!} u^{(n)}(0) \| \le \frac{C}{(n+1)!} \qquad \boxtimes$$

Theorem (Taylor's formula with integral remainder)

Let $f : U \overset{\text{open}}{\subset} X \longrightarrow Y$ be C^{n+1} map.

If the segment $[a, a+h]$ is contained in U, then:

$$f(a+h) - f(a) - Df(a)h - \cdots - \frac{1}{n!} D^n f(a) \underbrace{(h, \ldots, h)}_{n \text{ times}}$$

$$= \int_0^1 \frac{1}{n!} (1-t)^n D^{n+1} f(a+th) \underbrace{(h, \ldots, h)}_{n+1 \text{ times}}\, dt \qquad \boxtimes$$

THM (Taylor's formula with Lagrange's form of remainder)

Hypothesis in previous theorem weakened to

'f is $(n+1)$ times differentiable on U

and ~~there~~ a constant C which

majorizes $\|D^{n+1}f(x)\|$ $\forall\, x \in U$'

then:

$$\left\| f(a+h) - f(a) - Df(a)h - \cdots - \frac{1}{n!} D^n f(a)(\underbrace{h,\ldots,h}_{n\text{ times}}) \right\|$$

$$\leq \frac{C}{(n+1)!} \|h\|^{n+1} \qquad \blacksquare$$
