ENEE 664  Optimal Control  Homework 6
due March 15 to Semih Kara (GTA).

1. On page 4 of Lecture Notes 5 (a), in the EXERCISE, we assume continuity of the
first partial derivatives of \( f, \frac{\partial f}{\partial x} \).

Do we need the stronger notion of uniform continuity [read up on this concept]? 

2. In Theorem 2, Lecture Notes 5 (b), page 7, we show that,

\[
\text{if } \dim (X) = m < \infty \text{ and for } n \leq m, \text{ if } f, f_1, f_2, \ldots, f_n \text{ are linear functionals on } X \text{ and if for each } x \in X, \bigcap_{i=1}^{n} \ker (f_i) \subset \ker (f) \text{ then } f = \sum_{i=1}^{n} \alpha_i f_i \text{ for some } \alpha_i, i = 1, 2, \ldots, n.
\]

Remove the assumption that \( X \) is finite dimensional and prove the same result.

\text{Hint: } [\text{Consider } \tilde{f} = \sum_{x_i} f(x_i) f_i. \text{ Show that } \tilde{f} = f \text{ on the subspace } V = \text{span } \{ x_1, x_2, \ldots, x_n \} \text{ where } \{ x_i : i = 1, 2, \ldots, n \} \text{ satisfies } \delta_{ij} (x_i) = \delta_{ij} \text{ the Kronecker symbol. Extend this } \tilde{f} \text{ to all of } X \text{ so that } \tilde{f} = f \text{ on all of } X.]
3. Let \( X = \{ x(\cdot) : [0, 1] \rightarrow \mathbb{R}^2 \mid x(t) \text{ is continuously differentiable in } t \} \)
Let \( g : X \rightarrow \mathbb{R} \) be the nonlinear functional
\[
g(x) = \int_0^1 (x_1(t)x_2'(t) - x_2(t)x_1'(t)) dt
\]
where \( x_1(\cdot) \) and \( x_2(\cdot) \) are respectively the first and second components of \( x(\cdot) \).
Let \( \mathcal{F} = \{ x(\cdot) \in X \mid g(x) = 0; x_i(0) = x_i(1) = 0, i = 1, 2 \} \)

For a suitable norm on \( X \), determine conditions for \( x \) to be a regular point of \( \mathcal{F} \).

4. Given \( A, B > 0 \) and \( a, b \in \mathbb{R} \) and \( A \neq B \), find (the) point in \( \mathbb{R}^2 \) that is closest to the origin \((0, 0)\) and also lies on the ellipse
\[
\frac{(x-a)^2}{A} + \frac{(y-b)^2}{B} = 1.
\]

5. Consider \( f(K) = K^2 - L \) defined on \( n \times n \) symmetric matrices \( K \) and \( L \in \mathbb{R}^n \) given. It is of interest to solve \( f(K) = 0 \).
If \( L \) is positive semidefinite, extend the classical Newton algorithm to the setting when \( n = 2 \) and \( L = (a \ b; b \ c) \) and find \( K \).