When we seek to understand data via linear static models, the following representation holds:

\[ y = x\beta + \varepsilon \]

data

Here \( y \) is a \( n \times 1 \) vector, \( \varepsilon \) is a \( n \times 1 \) noise vector, \( \beta \) is a vector of parameters to be extracted from data, and each row of the \( n \times k \) matrix \( X \) is an instantiation of a regressor vector. Regressor vectors may arise from data outcomes of experiments or may be assumed models, e.g.

\[ x = (1, x_1, x_1^2, \ldots, x_1^{k-1}) \]

arises when we seek to represent \( y \) as a polynomial \( \sum_{i=1}^{k} \beta_i x_1^{i-1} \). To determine \( \hat{\beta} \), an estimate of \( \beta \), one might resort to least squares:

\[ \hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} (y_i - \sum_{j=1}^{k} x_1^{j-1} \beta_j)^2 \]

\[ = \arg\min_{\beta} (y - x\beta)^T (y - x\beta) \]
This idea may be extended to the setting of indirect adaptive control, where a controller is adapted using a running estimate of a parameter in a partially known model, with the aim of matching a known reference model.

\[ P: \quad \dot{y}_p = -a y_p + k_p u \quad \text{(plant)} \]

\[ M: \quad \dot{y}_m = -a y_m + r \quad \text{(ref. model)} \]

\( a > 0 \) is known but \( k_p \) is unknown.

**Ideal Controller**

\[ u(t) = \Theta^* r(t) \]

with \( \Theta^* = \frac{1}{k_p} \) if known.

**Adaptive Controller**

\[ u(t) = \Theta(t) r(t) \]

with \( \Theta(t) \) suitably adapted.

*Example:* \( \Theta(t) = \frac{1}{\tilde{w}(t)} \) (certainty equivalence)

where \( \tilde{w}(t) = \text{running estimate of parameter } k_p \).

**How do we get this?**

1. Feed \( u(t) \) to the process system

\[ R: \quad \dot{w}(t) = -a w(t) + u(t) \]
Let $\overline{\theta}^* = k_p$ denote true parameter. Then

$$\eta(t) = T(t) - \overline{\theta}^* = \text{parameter error}$$

$$\epsilon_p(t) = y_p(t) - y_m(t) = \text{output error}$$

$$\epsilon_i(t) = \overline{T}(t) w(t) - y_p(t) = \text{identification error}$$

(iii)
Because $a > 0$

If the plant and regressor have been running for a while then the effects of plant initial condition $y(0)$ and regressor initial condition $w(0)$ die out and we can write

$$y_p(t) = k_p w(t) = \overline{\theta}^* w(t)$$

$$\Rightarrow \quad \epsilon_i(t) = (T(t) - \overline{\theta}^*) w(t) = q(t) w(t)$$

Now we obtain $T(t)$ by the identification algorithm

$$\overline{T} = \left( I - \overline{\theta}^* \right) = \psi = -\gamma e_i w = -\gamma \psi w^2$$

with $\gamma > 0$
Let \( V = \frac{1}{2} \gamma^2 \) be a measure of error.

Along trajectories of the identification algorithm,

\[
\frac{dV}{dt} = \gamma \dot{\gamma} = -\gamma \gamma^2 \omega^2 \leq 0
\]

So identification algorithm has the property that it reduces the parameter estimation error. Does it reduce the error to zero?

Notice

\[
\gamma(t) = \gamma(0) \exp \left( -\alpha \int_0^t \omega^2(\sigma) d\sigma \right)
\]

If \( \int_0^t \omega^2(\sigma) d\sigma \rightarrow \infty \) as \( t \rightarrow \infty \) (condition of persistent excitation), then \( \gamma(t) \rightarrow 0 \) as \( t \rightarrow \infty \), and hence \( \gamma(t) \rightarrow \gamma^* = \text{true parameter value} \).

But this condition is dependent on what \( \gamma(\cdot) \) is fed to the sensor system.
Indirect Adaptive Control

Red loop: identification
Green loop: adaptive control

certainty equivalent.