Calculating the Electromagnetic Modes of Anisotropic Waveguides Using Finite Difference Method

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Abstract— **Numerical method is developed to calculate the electromagnetic modes of anisotropic waveguides using the finite difference method. Many materials used in integrated optical waveguides exhibit birefringence or anisotropy, yet most of the available numerical method is unable to accurately calculate the electromagnetic modes of anisotropic waveguides, especially in the general case when the principle axes of the material do not coincide with the waveguide axes. This project builds upon an earlier software package that handles the case of waveguides comprised of isotropic materials.**

I. Introduction

As optics is growing more rapidly in this era, more micro-structures are becoming optical in various fields of technology, from communications to sensing. More devices are built of anisotropic crystals to serve as optical waveguides; thus, the need for an efficient method of simulating these devices is more crucial. In order to make a more reliable design, more reliable methods are required to simulate various effects. Finite-difference method in solving for effective index of refraction has been proven to be one of the most efficient and accurate way to simulate waveguide. This paper is an improvement on such method to solve for electromagnetic modes of an anisotropic waveguide. Since most of the finite difference methods are only capable of simulating isotropic waveguide [1], [2]; changes have been made so that the method facilitates anisotropic materials as well.

II. Theory

Propagation of electromagnetic waves in dielectric waveguide depends intrinsically on size, dimensions, and the material that the waveguide is made of. Optical waveguides are made in form of different structures. One general common form of waveguide is ridge waveguide that is shown in figure 1. The structure, put to the computation window, is divided into grids in order to approximate differential operators.

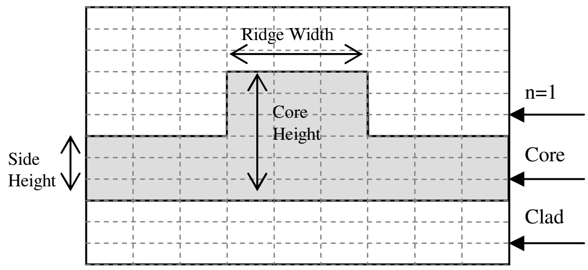

Fig. 1 – Discretization of a ridge

In some specific cases, the core could be made of an anisotropic crystal, such as Lithium Niobate. These materials have a different permittivity in the direction of each axis. This intrinsic property can be represented by a permittivity tensor in the following form:

$$\bar{\varepsilon} \equiv \begin{bmatrix} \varepsilon_{x'} & 0 & 0 \\ 0 & \varepsilon_{y'} & 0 \\ 0 & 0 & \varepsilon_{z'} \end{bmatrix} \tag{1}$$

x', y', and z' are the principle axes of the crystal. When a waveguide is made of such material, the axes of the material may not always align with the axes of the waveguide. Thus, the epsilon matrix in the following equation would have nine non-zero elements.

$$[D] = [\varepsilon] \cdot [E] \tag{2}$$

Nevertheless, in most of the cases, the longitudinal axis of the waveguide is lined up with one axis of the material (i.e. z' and z are aligned) in the construction of the waveguide. Therefore, epsilon matrix goes through a rotation around z-axis operation in (4).

$$R \equiv \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\bar{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} = R \cdot \begin{bmatrix} \varepsilon_{x'} & 0 & 0 \\ 0 & \varepsilon_{y'} & 0 \\ 0 & 0 & \varepsilon_{z'} \end{bmatrix} \cdot R^T \tag{4}$$

$\varepsilon_{xy} = \varepsilon_{yx}$ ($[\varepsilon]$ is Hermitian)

As a result, the equation for D will have the form of (5).

$$\begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \tag{5}$$

$$D_z = \varepsilon_{zz} E_z \tag{6}$$

In order to get the eigenmode equations for propagation of electromagnetic fields in the waveguide, the electric and magnetic fields have been assume to have the following form:

$$\vec{H}(x,y,z) = (\vec{H}_t + \hat{z}H_z)e^{-j\beta z} \tag{7}$$

$$\vec{E}(x,y,z) = (\vec{E}_t + \hat{z}E_z)e^{-j\beta z} \tag{8}$$

We can show that, by knowing only two components, H_x and H_y, all the components of electric and magnetic field can be derived from Maxwell equations:

$$H_z = \frac{-j}{\beta}\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right) \tag{9}$$

$$E_z = \frac{-j}{\omega\varepsilon_0\varepsilon_{zz}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \tag{10}$$

$$D_x = -\frac{1}{\omega\beta}\left(\frac{\partial^2 H_x}{\partial y \partial x} + \frac{\partial^2 H_y}{\partial y^2}\right) + \frac{\beta}{\omega} H_y \tag{11}$$

$$D_y = \frac{1}{\omega\beta}\left(\frac{\partial^2 H_y}{\partial x \partial y} + \frac{\partial^2 H_x}{\partial x^2}\right) - \frac{\beta}{\omega} H_x \tag{12}$$

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{\varepsilon_0\left(\varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^{\;2}\right)} \begin{bmatrix} \varepsilon_{yy} & -\varepsilon_{xy} \\ -\varepsilon_{xy} & \varepsilon_{xx} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} \tag{13}$$

With these assumptions, there would be two linearly polarized waves. From the transverse components of the equation (14) below, the wave equations (15) and (16) are obtained in the form of the eigensystem for the transverse magnetic field.

$$\nabla^2 H = -j\omega \nabla \times D \tag{14}$$

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\varepsilon_{yy}}{\varepsilon_{zz}}\frac{\partial^2 H_x}{\partial y^2} - \frac{\varepsilon_{xy}}{\varepsilon_{zz}}\frac{\partial^2 H_y}{\partial x^2} + \frac{\varepsilon_{xy}}{\varepsilon_{zz}}\frac{\partial^2 H_x}{\partial x \partial y} + (1 - \frac{\varepsilon_{yy}}{\varepsilon_{zz}})\frac{\partial^2 H_y}{\partial x \partial y} + k^2(\varepsilon_{yy}H_x - \varepsilon_{xy}H_y) = \beta^2 H_x \tag{15}$$

$$\frac{\partial^2 H_y}{\partial y^2} + \frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{\partial^2 H_y}{\partial x^2} - \frac{\varepsilon_{xy}}{\varepsilon_{zz}}\frac{\partial^2 H_x}{\partial y^2} + \frac{\varepsilon_{xy}}{\varepsilon_{zz}}\frac{\partial^2 H_y}{\partial x \partial y} + (1 - \frac{\varepsilon_{xx}}{\varepsilon_{zz}})\frac{\partial^2 H_x}{\partial x \partial y} + k^2(\varepsilon_{xx}H_y - \varepsilon_{xy}H_x) = \beta^2 H_y \tag{16}$$

The method essentially solves for β in (17) and thus the effective index of refraction in the direction of propagation.

$$\begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} = \beta^2 \begin{bmatrix} H_x \\ H_y \end{bmatrix} \tag{17}$$

$(n_{eff} = \beta\lambda/2\pi)$

A's are differential operators that are approximated by the finite-difference method from equation (15) and (16).

In regard of obtaining the finite-difference equations, an arbitrary point P in the mesh, to which the computation window is divided, is shown below with its neighboring points.

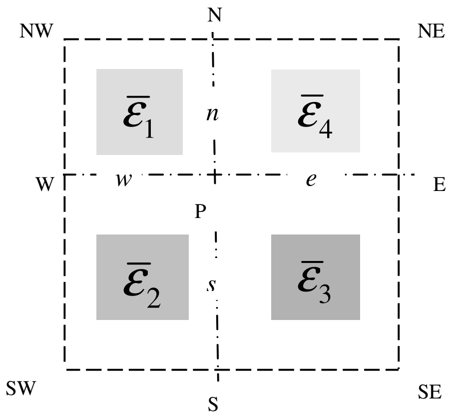

Fig. 2 – P and its neighboring point in a mesh

For any point P in the computation window, such grid could be constructed. Each of the four regions around the point is treated to have a different epsilon tensor. P is separated by distances n, e, s, and w from its neighboring points N, NE, E, SE, S, SW, W, and NW. The derivatives at the point P are approximated using these distances from the points around P.

We write the eigenequation for each of the four regions around P (I to IV, start with area NW, counter-clockwise). Four equations for H_x are shown below; the four equations for H_y are obtained similarly.

I - NW)

$$\begin{aligned} &\left(\frac{2}{w}\right)\frac{\partial H_x}{\partial x}\bigg|_W - \left(\frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I}\frac{2}{n}\right)\frac{\partial H_x}{\partial y}\bigg|_N - \left(\frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{2}{w}\right)\frac{\partial H_y}{\partial x}\bigg|_W + \left(\frac{2}{w^2} + \frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{1}{nw}\right)H_x^W + \left(\frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I}\frac{2}{n^2} + \frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{1}{nw}\right)H_x^N \\ &+\left(-\frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{2}{w^2} + (1 - \frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I})\frac{1}{nw}\right)H_y^W + \left((1 - \frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I})\frac{1}{nw}\right)H_y^N - \left(\frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{1}{nw}\right)H_x^{NW} - \left((1 - \frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I})\frac{1}{nw}\right)H_y^{NW} \\ &-\left(\frac{2}{w^2} + \frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I}\frac{2}{n^2} + \frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{1}{nw} - k^2\varepsilon_{yy}^I\right)H_x^P - \left(-\frac{\varepsilon_{xy}^I}{\varepsilon_{zz}^I}\frac{2}{w^2} + (1 - \frac{\varepsilon_{yy}^I}{\varepsilon_{zz}^I})\frac{1}{nw} + k^2\varepsilon_{xy}^I\right)H_y^P = \beta^2 H_x^P \end{aligned} \tag{18}$$

II - SW)

$$\left(\frac{2}{w}\right)\frac{\partial H_x}{\partial x}\bigg|_W + \left(\frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}}\frac{2}{s}\right)\frac{\partial H_x}{\partial y}\bigg|_S - \left(\frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{2}{w}\right)\frac{\partial H_y}{\partial x}\bigg|_W + \left(\frac{2}{w^2} - \frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{1}{sw}\right)H_x^W + \left(\frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}}\frac{2}{s^2} - \frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{1}{sw}\right)H_x^S$$
$$+\left(-\frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{2}{w^2} - (1-\frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}})\frac{1}{sw}\right)H_y^W - \left((1-\frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}})\frac{1}{sw}\right)H_y^S + \left(\frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{1}{sw}\right)H_x^{SW} + \left((1-\frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}})\frac{1}{sw}\right)H_y^{SW} \tag{19}$$
$$-\left(\frac{2}{w^2} + \frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}}\frac{2}{s^2} - \frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{1}{sw} - k^2\varepsilon_{yy}^{II}\right)H_x^P - \left(-\frac{\varepsilon_{xy}^{II}}{\varepsilon_{zz}^{II}}\frac{2}{w^2} - (1-\frac{\varepsilon_{yy}^{II}}{\varepsilon_{zz}^{II}})\frac{1}{sw} + k^2\varepsilon_{xy}^{II}\right)H_y^P = \beta^2 H_x^P$$

III - SE)

$$\left(\frac{-2}{e}\right)\frac{\partial H_x}{\partial x}\bigg|_E + \left(\frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}}\frac{2}{s}\right)\frac{\partial H_x}{\partial y}\bigg|_S + \left(\frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{2}{e}\right)\frac{\partial H_y}{\partial x}\bigg|_E + \left(\frac{2}{e^2} + \frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{1}{se}\right)H_x^E + \left(\frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}}\frac{2}{s^2} + \frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{1}{se}\right)H_x^S$$
$$+\left(-\frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{2}{e^2} + (1-\frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}})\frac{1}{se}\right)H_y^E + \left((1-\frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}})\frac{1}{se}\right)H_y^S - \left(\frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{1}{se}\right)H_x^{SE} - \left((1-\frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}})\frac{1}{se}\right)H_y^{SE} \tag{20}$$
$$-\left(\frac{2}{e^2} + \frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}}\frac{2}{s^2} + \frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{1}{se} - k^2\varepsilon_{yy}^{III}\right)H_x^P - \left(-\frac{\varepsilon_{xy}^{III}}{\varepsilon_{zz}^{III}}\frac{2}{e^2} + (1-\frac{\varepsilon_{yy}^{III}}{\varepsilon_{zz}^{III}})\frac{1}{se} + k^2\varepsilon_{xy}^{III}\right)H_y^P = \beta^2 H_x^P$$

IV - NE)

$$\left(\frac{-2}{e}\right)\frac{\partial H_x}{\partial x}\bigg|_E + \left(\frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{2}{n}\right)\frac{\partial H_x}{\partial y}\bigg|_N + \left(\frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{2}{e}\right)\frac{\partial H_y}{\partial x}\bigg|_E + \left(\frac{2}{e^2} - \frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{1}{ne}\right)H_x^E + \left(\frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{-2}{n^2} - \frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{1}{ne}\right)H_x^N$$
$$+\left(-\frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{2}{e^2} - (1-\frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}})\frac{1}{ne}\right)H_y^E - \left((1-\frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}})\frac{1}{ne}\right)H_y^N + \left(\frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{1}{ne}\right)H_x^{NE} + \left((1-\frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}})\frac{1}{ne}\right)H_y^{NE} \tag{21}$$
$$-\left(\frac{2}{e^2} - \frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{2}{n^2} - \frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{1}{ne} - k^2\varepsilon_{yy}^{IV}\right)H_x^P - \left(\frac{\varepsilon_{xy}^{IV}}{\varepsilon_{zz}^{IV}}\frac{2}{e^2} - (1-\frac{\varepsilon_{yy}^{IV}}{\varepsilon_{zz}^{IV}})\frac{1}{ne} + k^2\varepsilon_{xy}^{IV}\right)H_y^P = \beta^2 H_x^P$$

Using boundary conditions and appropriate finite-difference equivalent for derivatives, we reduce these eight equations to a two-by-two system. Regarding the boundary conditions, the longitudinal components of fields, H_z and E_z, are continuous at the horizontal and vertical boundaries [2]. They are calculated from the equations (22) and (23) of transverse components:

$$E_z = \frac{1}{j\varepsilon_{zz}k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y}\right) \tag{22}$$

$$H_z = \frac{1}{j\beta}\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right) \tag{23}$$

For every infinitesimally close point above and below the horizontal, boundary continuity of H_z gives

$$\frac{\partial H_y}{\partial y}\bigg|_n = \frac{\partial H_y}{\partial y}\bigg|_s \tag{24}$$

And, for every infinitesimally close point to right or left of the vertical boundary along the vertical boundary, it gives

$$\frac{\partial H_x}{\partial x}\bigg|_e = \frac{\partial H_x}{\partial x}\bigg|_w \tag{25}$$

Continuity of the E_z on the horizontal boundary gives

$$\varepsilon_{zz}^{IV/III}\frac{\partial H_y}{\partial x}\bigg|_W - \varepsilon_{zz}^{I/II}\frac{\partial H_y}{\partial x}\bigg|_E = \left(\varepsilon_{zz}^{IV/III} - \varepsilon_{zz}^{I/II}\right)\frac{\partial H_x}{\partial y} \tag{26}$$

and on the vertical boundary, it similarly gives

$$\varepsilon_{zz}^{I/IV}\frac{\partial H_x}{\partial y}\bigg|_S - \varepsilon_{zz}^{II/III}\frac{\partial H_x}{\partial y}\bigg|_N = \left(\varepsilon_{zz}^{I/IV} - \varepsilon_{zz}^{II/III}\right)\frac{\partial H_y}{\partial x} \tag{27}$$

These six boundary conditions reduce four equations (18) – (21) to one equation for H_x and similarly for H_y. Each of which would contain two derivatives that are then approximated by the following central finite-difference equivalents.

$$\frac{\partial H_y}{\partial x} = \frac{wH_y^E}{e(e+w)} + \frac{(e-w)H_y^P}{ew} - \frac{eH_y^W}{w(e+w)} \tag{28}$$

$$\frac{\partial H_x}{\partial y} = \frac{sH_x^N}{n(n+s)} + \frac{(n-s)H_x^P}{ns} - \frac{nH_x^S}{s(n+s)} \tag{29}$$

As a result, the values of the transverse magnetic field at any point P would be written in terms of the value of the field at the neighboring points. In other words, all the coefficients of the A's in (17) would become numerical values. The coefficient of H^P and the field at every neighboring point around P is shown in the appendix in order to avoid lengthy equations in the paper.

III. Results

As mentioned above, one of the axes of the material is assumed to be in the same direction as the direction of propagation. And, the other two axes are rotated through an angle θ to indicate the effect of an anisotropic material. Due to the non-symmetric property of the material, the whole structure of the waveguide is simulated. For all of our simulations here, the wavelength used is 1.55 µm.

One of the results is compared with the result obtained from Lusse [3]. The effective refractive index of the waveguide with width 3 µm and height 2µm, isotropic cladding n_c = 3.4 and an anisotropic material $n_{x'}$ = $n_{z'}$ = 3.5, $n_{y'}$ =3.45 is 3.48063 under the grid size 250 by 250 (50 points per micron), which agrees with Lusse's result.

Fig. 3 shows the cross-section of a square waveguide with dimension 2µm. The material is uniaxial $LiNbO_3$ with $n_{x'}$ = 2.20, $n_{y'}$ = $n_{z'}$ = 2.29 [4]. The medium outside the waveguide is air and no rotation is made in this case. The simulated effective refractive index in this case is 2.23178. 250 by 250 grid is used (50 points per micron). The contour lines in figure 3 represent the contour 3-dB fall diagram.

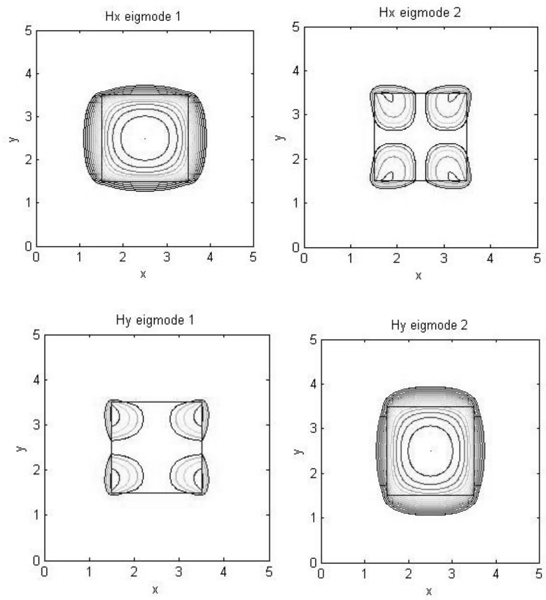

Fig. 3 – Cross-section of a square waveguide with dimension 2µm with anisotropic material n_x = 2.20, n_y = n_z = 2.29

Fig. 4 shows exactly the same dimension and material with a rotation 45° around z-axis. The simulated effective refractive index is 2.18661. The grid used in the program is also 50 points per micron. The results show a non-symmetric magnetic wave inside the waveguide.

With the above waveguide with n_x = 2.20, n_y = n_z = 2.29, the program is repeated using different angles. The point mesh is reduced to 20 per micron. The result is tabulate in Table 1.The result show a decrease of effective refractive index from 0° to 45°. Figure 6 shows the cross-section of the fundamental mode of the waveguide corresponding to each angle change in table 1.

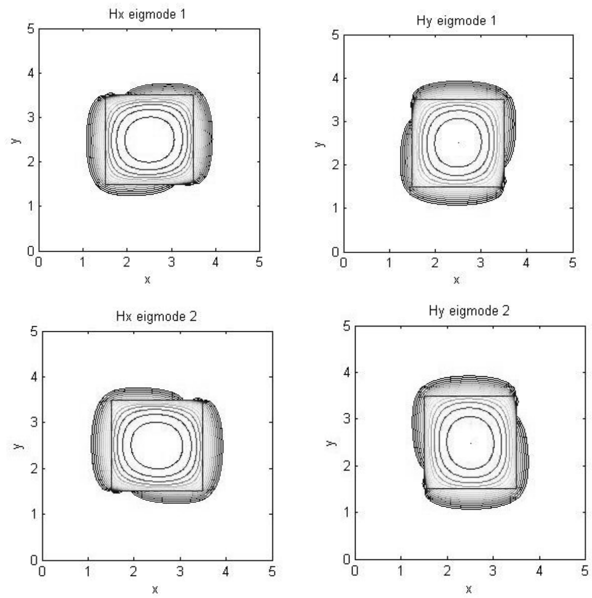

Fig. 4 – Cross-section of a square waveguide with dimension 2µm with anisotropic material n_x = 2.20, n_y = n_z = 2.29 and a rotations angle of 45°

angle (degree)	n_{eff}
0	2.23167
9	2.22949
18	2.22313
27	2.21324
36	2.20078
45	2.18740
54	2.20078
63	2.21324
72	2.22313
81	2.22949
90	2.23167

Table 1 – Effective refractive index with different angle of rotation in x-y plane

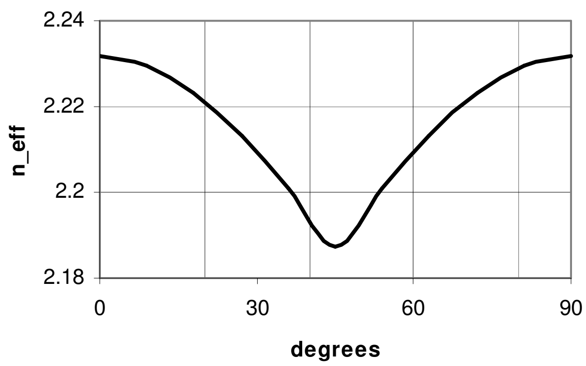

Fig. 5 – Effective refractive index vs. relative angle of x-y plane

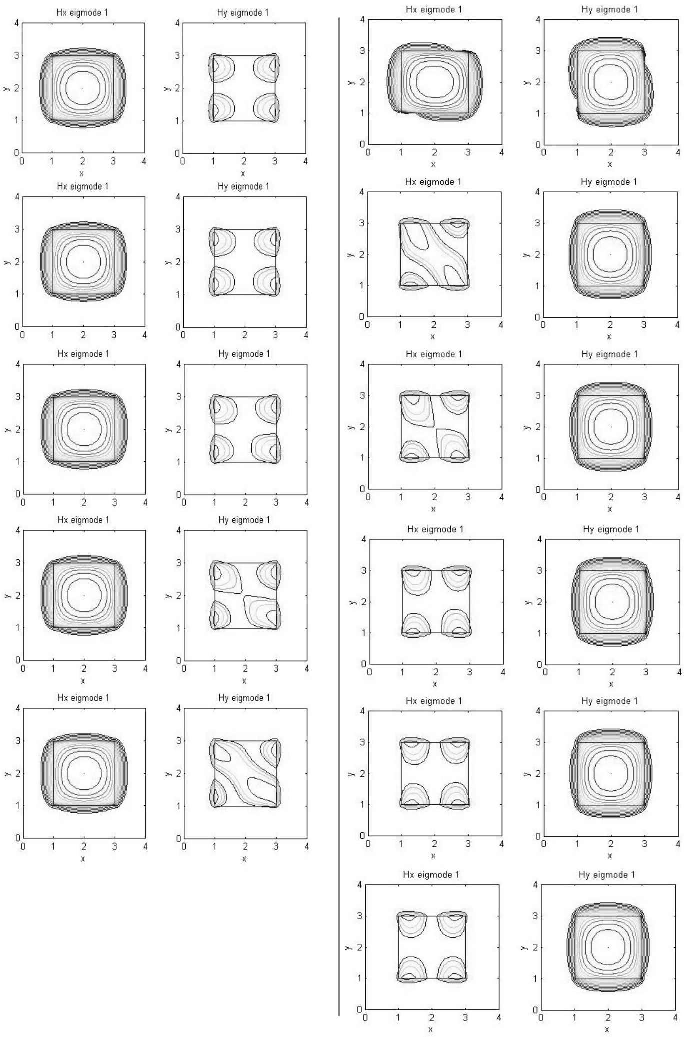

Fig. 6 – Cross-section of an anisotropic waveguide with a rotation from 0º to 90º

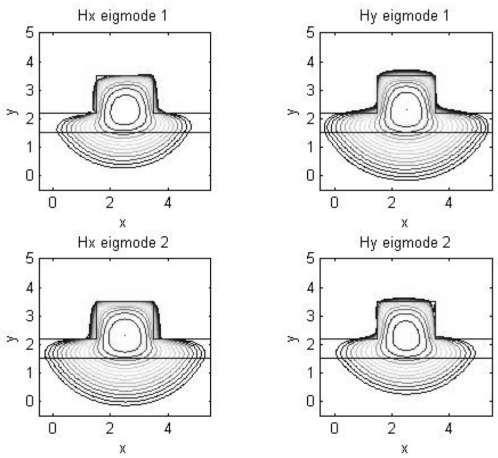

Fig. 7 – Cross-section of a ridge waveguide with anisotropic material $n_{x'}$ = 3.40, $n_{y'}$ = 3.45 and $n_{z'}$ = 3.50. An isotropic cladding below it where n_c = 3.34 and air above the waveguide

A ridge waveguide is also simulated with core height of 2.0 µm, side height of 0.7 µm, and ridge width of 2.0 µm. Also, there are 20 mesh points per micron in computation window. The waveguide is anisotropic with $n_{x'}$ = 3.40, $n_{y'}$ = 3.45 and $n_{z'}$ = 3.50. The waveguide is cover by air and the cladding below is isotropic with n_c = 3.44. The effective refractive index is 3.39237. Figure 7 shows the cross-section of the waveguide. A non-symmetric magnetic wave is again shown inside the waveguide.

IV. Conclusion

In conclusion, a more general way of simulating waveguide is implemented for anisotropic materials. Under the assumption that one of the axis of the material is pointing towards the direction of propagation, the method is capable of resolving the effective refractive index of an anisotropic material by calculating its eigenmode using finite difference method. The results of the effective refractive index of an anisotropic material with 0 degree rotations are compared to the previous published data. Anisotropic materials with rotated axes are also simulated. With this routine, different waveguide with an anisotropic material can be simulated.

Acknowledgment

At this point we feel truly obligated to use this opportunity to thank those who helped us in achieving our goal. First of all, we are in great debt to Dr. Murphy, who has always been kind enough to enlighten us in moments of doubt and guide us to our track. Then, we thank Pavin Apiratikul, and the others in the Photonics Research Laboratory in the University of Maryland, who provided the background for our research. And, last but not least, we should recognize the wonderful effort of Dr. Judith Bell, Ms. India Tiller, and the entire fine personnel of the MERIT program in the department of electrical and computer engineering at the University of Maryland, who made this program possible for us.

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APPENDIX

The finite different eigenequations and its coefficients are presented in this appendix:

Equation for H_x:

$$a_{xx}{}^{W}H_x{}^{W}+a_{xx}{}^{E}H_x{}^{E}+a_{xx}{}^{N}H_x{}^{N}+a_{xx}{}^{S}H_x{}^{S}+a_{xx}{}^{P}H_x{}^{P}+a_{xx}{}^{SW}H_x{}^{SW}+a_{xx}{}^{NW}H_x{}^{NW}+a_{xx}{}^{NE}H_x{}^{NE}+a_{xx}{}^{SE}H_x{}^{SE}$$
$$+a_{xy}{}^{W}H_y{}^{W}+a_{xy}{}^{E}H_y{}^{E}+a_{xy}{}^{N}H_y{}^{N}+a_{xy}{}^{S}H_y{}^{S}+a_{xy}{}^{P}H_y{}^{P}+a_{xy}{}^{SW}H_y{}^{SW}+a_{xy}{}^{NW}H_y{}^{NW}+a_{xy}{}^{NE}H_y{}^{NE}+a_{xy}{}^{SE}H_y{}^{SE}$$
$$=\beta^2 H_x{}^{p}$$

Equation for H_y:

$$a_{yx}{}^{W}H_x{}^{W}+a_{yx}{}^{E}H_x{}^{E}+a_{yx}{}^{N}H_x{}^{N}+a_{yx}{}^{S}H_x{}^{S}+a_{yx}{}^{P}H_x{}^{P}+a_{yx}{}^{SW}H_x{}^{SW}+a_{yx}{}^{NW}H_x{}^{NW}+a_{yx}{}^{NE}H_x{}^{NE}+a_{yx}{}^{SE}H_x{}^{SE}$$
$$+a_{yy}{}^{W}H_y{}^{W}+a_{yy}{}^{E}H_y{}^{E}+a_{yy}{}^{N}H_y{}^{N}+a_{yy}{}^{S}H_y{}^{S}+a_{yy}{}^{P}H_y{}^{P}+a_{yy}{}^{SW}H_y{}^{SW}+a_{yy}{}^{NW}H_y{}^{NW}+a_{yy}{}^{NE}H_y{}^{NE}+a_{yy}{}^{SE}H_y{}^{SE}$$
$$=\beta^2 H_y{}^{p}$$

We define R_y and R_x just to make the coefficients easier to read:

$$R_y=(\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})+(\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}})$$

$$R_x=(\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}})(\frac{n}{e_{yy1}}+\frac{s}{e_{yy2}})+(\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}})$$

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Equation for Hx - coefficient of Hx:
axxw =
```

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{n}{e_{yy1}}(\frac{2}{w^2}+\frac{e_{xy1}}{(e_{zz1})(n)(w)})+\frac{s}{e_{yy2}}(\frac{2}{w^2}-\frac{e_{xy2}}{(e_{zz2})(s)(w)})))$$

```
axxe =
```

$$(\frac{1}{R_x})(((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy3}}(\frac{2}{e^2}+\frac{e_{xy3}}{(e_{zz3})(s)(e)})+\frac{n}{e_{yy4}}(\frac{2}{e^2}-\frac{e_{xy4}}{(e_{zz4})(n)(e)})))$$

```
axxs =
```

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{s}{e_{yy2}}((\frac{2}{s^2})(\frac{e_{yy2}}{e_{zz2}})-(\frac{e_{xy2}}{e_{zz2}(s)(w)})))+((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy3}}((\frac{2}{s^2})(\frac{e_{yy3}}{e_{zz3}})+\frac{e_{xy3}}{e_{zz3}(s)(e)})))$$

```
axxn =
```

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{n}{e_{yy1}}((\frac{2}{n^2})(\frac{e_{yy1}}{e_{zz1}})+(\frac{e_{xy1}}{e_{zz1}(n)(w)})))+((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy4}}((\frac{2}{n^2})(\frac{e_{yy4}}{e_{zz4}})-\frac{e_{xy4}}{e_{zz4}(n)(e)})))$$

```
axxnw =
```

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{n}{e_{yy1}}(-\frac{e_{xy1}}{(e_{zz1})(n)(w)})))$$

```
axxne =
```

$$(\frac{1}{R_x})(((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{n}{e_{yy4}}(\frac{e_{xy4}}{(e_{zz4})(n)(e)})))$$

```
axxse =
```

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy3}}(-\frac{e_{xy3}}{(e_{zz3})(s)(e)})))$$

```
axxsw =
```

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{s}{e_{yy2}}(\frac{e_{xy2}}{(e_{zz2})(s)(w)})))$$

axxp =

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{n}{e_{yy1}}(\frac{-2}{w^2}-(\frac{2}{n^2})(\frac{e_{yy1}}{e_{zz1}})-\frac{e_{xy1}}{e_{zz1}(n)(w)})+k^2e_{yy1})\ldots$$

$$+\frac{s}{e_{yy2}}(-\frac{2}{w^2}-(\frac{2}{s^2})(\frac{e_{yy2}}{e_{zz2}})+(\frac{e_{xy2}}{e_{zz2}(s)(w)})+k^2e_{yy2}))\ldots$$

$$+(\frac{1}{w}(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy3}}((\frac{-2}{e^2})-(\frac{2}{s^2})(\frac{e_{yy3}}{e_{zz3}})-\frac{e_{xy3}}{e_{zz3}(s)(e)}+k^2e_{yy3})+\frac{n}{e_{yy4}}(\frac{-2}{e^2}-(\frac{2}{n^2})(\frac{e_{yy4}}{e_{zz4}})+\frac{e_{xy4}}{e_{zz4}(n)(e)})+k^2e_{yy4})))$$

Equation for Hx - coefficient of Hy:

axyw = $$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{n}{e_{yy1}}(-2(\frac{e_{xy1}}{e_{zz1}(w^2)})+(1-(\frac{e_{yy1}}{e_{zz1}}))(\frac{1}{(n)(w)}))+\frac{s}{e_{yy2}}(-2(\frac{e_{xy2}}{e_{zz2}(w^2)})-(1-(\frac{e_{yy2}}{e_{zz2}}))(\frac{1}{(s)(w)})))$$

$$+((\frac{2}{e_{zz2}e_{zz1}e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}})((e_{zz1}-e_{zz2})-(\frac{e_{xy2}}{e_{yy2}})(e_{zz1})(\frac{s}{w})-(\frac{e_{xy1}}{e_{yy1}})(e_{zz2})(\frac{n}{w}))+(\frac{2}{e_{zz3}e_{zz4}w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}})((e_{zz4}-e_{zz3})$$

$$+(\frac{e_{xy3}}{e_{yy3}})(e_{zz4})(\frac{s}{e})-(\frac{e_{xy4}}{e_{yy4}})(e_{zz3})(\frac{n}{e})))(\frac{-e}{w(e+w)}))$$

axye =

$$(\frac{1}{R_x})(((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy3}}(-2(\frac{e_{xy3}}{e_{zz3}(e^2)})+(1-(\frac{e_{yy3}}{e_{zz3}}))(\frac{1}{(s)(e)}))+\frac{n}{e_{yy4}}(-2(\frac{e_{xy4}}{e_{zz4}(e^2)})-(1-(\frac{e_{yy4}}{e_{zz4}}))(\frac{1}{(n)(e)})))\cdots$$

$$+((\frac{2}{e_{zz2}e_{zz1}e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}})((e_{zz1}-e_{zz2})-(\frac{e_{xy2}}{e_{yy2}})(e_{zz1})(\frac{s}{w})-(\frac{e_{xy1}}{e_{yy1}})(e_{zz2})(\frac{n}{w}))+(\frac{2}{e_{zz3}e_{zz4}w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}})((e_{zz4}-e_{zz3})\cdots$$

$$+(\frac{e_{xy3}}{e_{yy3}})(e_{zz4})(\frac{s}{e})-(\frac{e_{xy4}}{e_{yy4}})(e_{zz3})(\frac{n}{e})))(\frac{w}{e(e+w)}))$$

axys =

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))((\frac{s}{e_{yy2}})((1-\frac{e_{yy2}}{e_{zz2}})(-\frac{1}{sw})))+(\frac{1}{w}(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))((\frac{s}{e_{yy3}})((1-\frac{e_{yy3}}{e_{zz3}})(\frac{1}{ne}))))$$

axyn = $$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))((\frac{n}{e_{yy1}})((1-\frac{e_{yy1}}{e_{zz1}})(\frac{1}{nw})))+(\frac{1}{w}(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))((\frac{n}{e_{yy4}})((1-\frac{e_{yy4}}{e_{zz4}})(\frac{-1}{ne}))))$$

axynw =

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{n}{e_{yy1}}(-(1-\frac{e_{yy1}}{e_{zz1}})\frac{1}{(n)(w)})))$$

axyne =

$$(\frac{1}{R_x})(((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{n}{e_{yy4}}((1-\frac{e_{yy4}}{e_{zz4}})\frac{1}{(n)(e)})))$$

axyse =

$$(\frac{1}{R_x})(((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))(\frac{s}{e_{yy3}}(-(1-\frac{e_{yy3}}{e_{zz3}})\frac{1}{(s)(e)})))$$

axysw =

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}}))(\frac{s}{e_{yy2}}((1-\frac{e_{yy2}}{e_{zz2}})\frac{1}{(s)(w)})))$$

axyp =

$$(\frac{1}{R_x})(((\frac{1}{e})(\frac{s}{e_{eyy3}}+\frac{n}{e_{yy4}}))((\frac{n}{e_{yy1}})(2(\frac{e_{xy1}}{e_{zz1}w^2})-(1-(\frac{e_{yy1}}{e_{zz1}}))(\frac{1}{nw})-k^2e_{xy1})+(\frac{s}{e_{yy2}})(2(\frac{e_{xy2}}{e_{zz2}(w^2)})+(1-\frac{e_{yy2}}{e_{zz2}})(\frac{1}{sw})$$
$$-k^2e_{xy2}))+((\frac{1}{w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}}))((\frac{s}{e_{yy3}})(2(\frac{e_{xy3}}{e_{zz3}e^2})-(1-(\frac{e_{yy3}}{e_{zz3}}))(\frac{1}{se})-k^2e_{xy3}+$$
$$(\frac{n}{e_{yy4}})(2(\frac{e_{xy4}}{e_{zz4}e^2})+(1-(\frac{e_{yy4}}{e_{zz4}}))(\frac{1}{ne})-k^2e_{xy4}))+((\frac{2}{e_{zz2}e_{zz1}e})(\frac{s}{e_{yy3}}+\frac{n}{e_{yy4}})((e_{zz1}-e_{zz2})-(\frac{e_{xy2}}{e_{yy2}})(e_{zz1})(\frac{s}{w})$$
$$-(\frac{e_{xy1}}{e_{yy1}})(e_{zz2})(\frac{n}{w}))+(\frac{2}{e_{zz3}e_{zz4}w})(\frac{s}{e_{yy2}}+\frac{n}{e_{yy1}})((e_{zz4}-e_{zz3})+(\frac{e_{xy3}}{e_{yy3}})(e_{zz4})(\frac{s}{e})-(\frac{e_{xy4}}{e_{yy4}})(e_{zz3})(\frac{n}{e})))(\frac{e-w}{ew})))$$

Equation for Hy - coefficient of Hy:

ayyw =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))(\frac{w}{e_{xx2}}((\frac{2}{w^2})(\frac{e_{xx2}}{e_{zz2}})-\frac{e_{xy2}}{e_{zz2}(s)(w)}))+((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))((\frac{w}{e_{xx1}})((\frac{2}{w^2})(\frac{e_{xx1}}{e_{zz1}})-\frac{e_{xy1}}{e_{zz1}(n)(w)})))$$

ayye =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))(\frac{e}{e_{xx3}}((\frac{2}{e^2})(\frac{e_{xx3}}{e_{zz3}})+\frac{e_{xy3}}{e_{zz3}(s)(e)}))+((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))((\frac{e}{e_{xx4}})((\frac{2}{e^2})(\frac{e_{xx4}}{e_{zz4}})-\frac{e_{xy4}}{e_{zz4}(n)(e)})))$$

ayys =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))((\frac{w}{e_{xx2}})(\frac{2}{s^2}+\frac{e_{xy2}}{e_{zz2}(s)(w)})+(\frac{e}{e_{xx3}})(\frac{2}{s^2}-(\frac{e_{xy4}}{e_{zz3}(s)(e)}))))$$

ayyn =

$$(\frac{1}{R_y})(((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))((\frac{w}{e_{xx1}})(\frac{2}{n^2}+\frac{e_{xy1}}{e_{zz1}(n)(w)})+(\frac{e}{e_{xx4}})(\frac{2}{n^2}-(\frac{e_{xy4}}{e_{zz4}(n)(e)}))))$$

ayynw =

$$(\frac{1}{R_y})(((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))((\frac{w}{e_{xx1}})(-\frac{e_{xy1}}{e_{zz1}(n)(w)})))$$

ayyne =

$$(\frac{1}{R_y})(((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))((\frac{e}{e_{xx4}})(\frac{e_{xy4}}{e_{zz4}(n)(e)})))$$

ayyse =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))((\frac{e}{e_{xx3}})(-\frac{e_{xy3}}{e_{zz3}(s)(e)})))$$

ayysw =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))((\frac{w}{e_{xx2}})(\frac{e_{xy2}}{e_{zz2}(s)(w)})))$$

ayyp =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))(\frac{w}{e_{xx2}}(\frac{-2}{s^2}-(\frac{2}{w^2})(\frac{e_{xx2}}{e_{zz2}})+\frac{e_{xy2}}{e_{zz2}(s)(w)})+k^2e_{xx2})\ldots$$
$$+\frac{e}{e_{xx3}}(-\frac{2}{s^2}-(\frac{2}{e^2})(\frac{e_{xx3}}{e_{zz3}})-(\frac{e_{xy3}}{e_{zz3}(s)(e)})+k^2e_{xx3}))\ldots$$
$$+(\frac{1}{s}(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))(\frac{w}{e_{xx1}}((\frac{-2}{n^2})-(\frac{2}{w^2})(\frac{e_{xx1}}{e_{zz1}})-\frac{e_{xy1}}{e_{zz1}(n)(w)}+k^2e_{xx1})+\frac{e}{e_{xx4}}(\frac{-2}{n^2}-(\frac{2}{e^2})(\frac{e_{xx4}}{e_{zz4}})+\frac{e_{xy4}}{e_{zz4}(n)(e)})+k^2e_{xx4})))$$

Equation for Hy - coefficient of Hx:

ayxw =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))((\frac{w}{e_{xx2}})(1-\frac{e_{xx2}}{e_{zz2}})(\frac{-1}{(s)(w)}))+(\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})(\frac{w}{e_{xx1}})(1-\frac{e_{xx1}}{e_{zz1}})(\frac{1}{(n)(w)}))$$

ayxe =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))((\frac{e}{e_{xx3}})(1-\frac{e_{xx3}}{e_{zz3}})(\frac{1}{(s)(e)}))+(\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})(\frac{e}{e_{xx4}})(1-\frac{e_{xx4}}{e_{zz4}})(-\frac{1}{(n)(e)}))$$

ayxs =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}}))(\frac{w}{e_{xx2}}(-2(\frac{e_{xy2}}{e_{zz2}(s^2)})-(1-(\frac{e_{xx2}}{e_{zz2}}))(\frac{1}{(s)(w)})+\frac{e}{e_{xx3}}(-2(\frac{e_{xy3}}{e_{zz3}(s^2)})+(1-(\frac{e_{xx3}}{e_{zz3}}))(\frac{1}{(s)(e)}))) \cdots$$

$$+((\frac{2}{e_{zz2}e_{zz3}n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}})((e_{zz3}-e_{zz2})-(\frac{e_{xy2}}{e_{xx2}})(e_{zz3})(\frac{w}{s})-(\frac{e_{xy3}}{e_{xx3}})(e_{zz2})(\frac{e}{s}))+(\frac{2}{e_{zz1}e_{zz4}s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})((e_{zz4}-e_{zz1}) \cdots$$

$$-(\frac{e_{xy1}}{e_{xx1}})(e_{zz4})(\frac{w}{n})-(\frac{e_{xy4}}{e_{xx4}})(e_{zz1})(\frac{e}{n})))(\frac{-n}{s(n+s)}))$$

ayxn =

$$(\frac{1}{R_y})(((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))(\frac{w}{e_{xx1}}(-2(\frac{e_{xy1}}{e_{zz1}(n^2)})+(1-(\frac{e_{xx1}}{e_{zz1}}))(\frac{1}{(n)(w)})+\frac{e}{e_{xx4}}(-2(\frac{e_{xy4}}{e_{zz4}(n^2)})-(1-(\frac{e_{xx4}}{e_{zz4}}))(\frac{1}{(n)(e)})))$$

$$+((\frac{2}{e_{zz2}e_{zz3}n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}})((e_{zz3}-e_{zz2})-(\frac{e_{xy2}}{e_{xx2}})(e_{zz3})(\frac{w}{s})-(\frac{e_{xy3}}{e_{xx3}})(e_{zz2})(\frac{e}{s}))+(\frac{2}{e_{zz1}e_{zz4}s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})((e_{zz4}-e_{zz1})$$

$$-(\frac{e_{xy1}}{e_{xx1}})(e_{zz4})(\frac{w}{n})-(\frac{e_{xy4}}{e_{xx4}})(e_{zz1})(\frac{e}{n})))(\frac{s}{n(n+s)}))$$

ayxnw =

$$(\frac{1}{R_y})(((\frac{1}{s})(\frac{w}{e_{exx2}}+\frac{e}{e_{xx3}}))((\frac{w}{e_{xx1}})(-(1-\frac{e_{xx1}}{e_{zz1}})(\frac{1}{(n)(w)}))))$$

ayxne =

$$(\frac{1}{R_y})(((\frac{1}{s})(\frac{w}{e_{exx2}}+\frac{e}{e_{xx3}}))((\frac{w}{e_{xx4}})((1-\frac{e_{xx4}}{e_{zz4}})(\frac{1}{(n)(e)}))))$$

ayxse =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{exx1}}+\frac{e}{e_{xx4}}))((\frac{e}{e_{xx3}})(-(1-\frac{e_{xx3}}{e_{zz3}})(\frac{1}{(s)(e)}))))$$

ayxsw =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{exx1}}+\frac{e}{e_{xx4}}))((\frac{w}{e_{xx2}})((1-\frac{e_{xx2}}{e_{zz2}})(\frac{1}{(s)(w)}))))$$

ayxp =

$$(\frac{1}{R_y})(((\frac{1}{n})(\frac{w}{e_{exx1}}+\frac{e}{e_{xx4}}))((\frac{w}{e_{xx2}})(2(\frac{e_{xy2}}{e_{zz2}s^2})+(1-(\frac{e_{xx2}}{e_{zz2}}))(\frac{1}{(s)(w)})-k^2e_{xy2})+(\frac{e}{e_{xx3}})(2(\frac{e_{xy3}}{e_{zz3}(s^2)})-(1-\frac{e_{xx3}}{e_{zz3}})(\frac{1}{(s)(e)})$$

$$-k^2e_{xy3}))+((\frac{1}{s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}}))((\frac{w}{e_{xx1}})(2(\frac{e_{xy1}}{e_{zz1}n^2})-(1-(\frac{e_{xx1}}{e_{zz1}}))(\frac{1}{nw})-k^2e_{xy1}+\ldots$$

$$(\frac{e}{e_{xx4}})(2(\frac{e_{xy4}}{e_{zz4}n^2})+(1-(\frac{e_{xx4}}{e_{zz4}}))(\frac{1}{ne})-k^2e_{xy4}))+((\frac{2}{e_{zz2}e_{zz3}n})(\frac{w}{e_{xx1}}+\frac{e}{e_{xx4}})((e_{zz3}-e_{zz2})-(\frac{e_{xy2}}{e_{xx2}})(e_{zz3})(\frac{w}{s})$$

$$-(\frac{e_{xy3}}{e_{xx3}})(e_{zz2})(\frac{e}{s}))+(\frac{2}{e_{zz1}e_{zz4}s})(\frac{w}{e_{xx2}}+\frac{e}{e_{xx3}})((e_{zz4}-e_{zz1})-(\frac{e_{xy1}}{e_{xx1}})(e_{zz4})(\frac{w}{n})-(\frac{e_{xy4}}{e_{xx4}})(e_{zz1})(\frac{e}{n})))(\frac{n-s}{ns})))$$