Basic Mathematics — Ph.D. Qualifying Examination Fall 2016

1. (7 pts.)

(1a) Find the inverse of the matrix $\begin{pmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$. Be careful to show your work. **(3 pts.)**

(1b) If $S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ show that $SS^T = I$ (i.e. that S is orthogonal). **(1 pt.)**

Also show that if $P = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, then SPS^T is diagonal. What are the eigenvalues and eigenvectors of P? **(3 pts.)**

2. (6 pts.) Solve the equation **a**x+**b**y+**c**z = **d** for the variables x, y, and z , where **a, b, c,** and **d** are real, constant vectors in three-dimensional space. Each variable should be written in terms of vectors **a**, **b**, **c**, and **d** using suitable vector operations.

3. (7 pts.) Solve the second order differential equation $2\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 3x = 5\cos(3t)$ subject to the boundary conditions at $t = 0 : x = 0$, $\frac{dx}{dt} = 1$.

Solutions

(1) Find inverse of $A := \begin{pmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ Using Cramer's rule

Determinant is $\det := 2 \cdot 2 - 4 \cdot (-2)$ $\det = 12$

Matrix of cofactors is $\begin{pmatrix} 4 & 2 & 0 \\ -8 & 2 & 0 \\ 8 & -2 & 6 \end{pmatrix}$

Transpose is $\begin{pmatrix} 4 & -8 & 8 \\ 2 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

Inverse is $B := \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{-1}{6} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ $A \cdot B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$$

$M := \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

$ev := \text{eigenvals}(M)$

$ev = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$evec1 := \text{eigenvec}(M, 4)$

$evec1 = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$

$evec2 := \text{eigenvec}(M, -2)$

$evec2 = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$

(2) Consider scalar triple product (**a**x+**b**y+**c**z).(**b**x**c**)=**d**.(**b**x**c**)

Since **b**.(**b**x**c**)=**c**.(**b**x**c**)=0 x**a**.(**b**x**c**)=**d**.(**b**x**c**), which gives x=**d.(bxc)/a.(bxc)**

To get y use scalar triple product with **axc.** **a.(axc)=c.(axc)=0** so **y**=**d.(axc)/b.(axc)**

To get c use scalar triple product with **axb** **z=d.(axb)/c.(axb)**

(3) The auxiliary equation is $2D^2 + 7D + 3 = 0$ with solution D=-1/2 and D=-3

General solution is $x_g(t) = Ae^{-t/2} + Be^{-3t}$

Particular solution must be of the form $x_p(t) = C\sin(3t) + D\cos(3t)$

Substitute in differential equation and compare sine and cosine terms gives

$-15C - 21D = 0$ and $-15D + 21C = 5$ giving C=0.158, D=-0.113.

Full solution is $x(t) = Ae^{-t/2} + Be^{-3t} + 0.158\sin(3t) - 0.113\cos(3t)$

Substituting initial conditions A+B-0.113=0, -0.5A-3B+0.474=1 gives A=0.346, B=-0.223