Linear Signals and Systems — Ph.D. Qualifying Exam Fall 2016

Part 1 (7 pts.)

The frequency response of a *discrete-time* linear time-invariant system is given over the frequency interval $(-\pi, \pi]$ by

$$H(e^{j\omega}) \;=\; \begin{cases} 1, & |\omega| \leq \pi/3 \\ 0, & \pi/3 < |\omega| \leq \pi \end{cases}$$

(1a) (3 pts.) Determine the system's impulse response sequence $h[n]$, expressing your answer in terms of real-valued quantities.

(1b) (4 pts.) If the input to the filter is given at all times $n \in \mathbf{Z}$ by

$$x[n] \;=\; \frac{\sin(n\pi/2)}{n\pi} + \sin\left(\frac{2\pi n}{7}\right) + \cos\left(\frac{3\pi n}{7}\right) \;,$$

determine the output sequence $y[n]$.

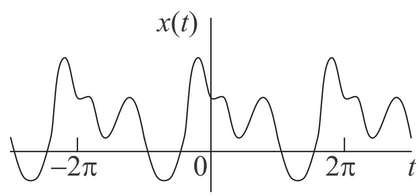
Part 2 (6 pts.)

A *continuous-time* linear time-invariant system has impulse response

$$h(t) \;=\; e^{-rt}u(t) \;,$$

where $r > 0$. (Here, and in Part 3 below, $u(t)$ is a unit step at $t = 0$.)

The input to the filter is a periodic signal $x(t)$ having period equal to 2π seconds, as shown below.



If $x(t)$ is given by the (convergent) sum

$$x(t) \;=\; \sum_{k=0}^{\infty} A_k \cos(kt + \phi_k) \;,$$

obtain a similar equation for the output $y(t)$ using the parameters $\{A_k\}$ and $\{\phi_k\}$.

Part 3 (7 pts.)

A *continuous-time*, *BIBO stable* linear time-invariant system has transfer function

$$H(s) \;=\; \frac{s}{4 - s^2}$$

over an appropriate region of convergence.

(3a) (2 pts.) What is the region of convergence of $H(s)$? Is the system causal?

(3b) (5 pts.) If the input to the system is $x(t) \;=\; u(t-1)$, determine the output $y(t)$.

Part 1

(1a)

(2*pi)*h[n] is given by the integral of H(exp(j*w))*exp(j*w*n)
over w in (-pi,pi]. Thus

(2*pi)*h[n] = (exp(j*pi/3) - exp(-j*pi/3))/(j*n)

i.e,
h[n] = sin(n*pi/3)/(n*pi)

(1b)
The system is an ideal lowpass filter with cutoff at w = pi/3,
gain = 1 and zero delay.

Since 2/7 < 1/3 < 3/7, it follows that the input

x'[n] = sin(2*pi*n/7) + cos(3*pi*n/7)

produces the output

y'[n] = sin(2*pi*n/7)

The remaining component of x[n], namely

x[n] - x'[n] = sin(n*pi/2)/(n*pi) = (say) g[n]

is the impulse response of a (similar) lowpass filter
with cutoff at w = pi/2. Since

G(exp(j*w))H(exp(j*w)) = H(exp(j*w)) ,

it follows that

y[n] = h[n] + y'[n] = sin(n*pi/3)/(n*pi) + sin(2*pi*n/7)

Part 2

h(t) = exp(-r*t)*u(t) has Fourier transform

H(j*W) = 1/(r + j*W) = (r - j*W)/(r^2 + W^2)

It follows that the input

cos(W*t + q)

produces the output

c*cos(W*t + q - arctan(W/r)) ,

where

  c = 1/sqrt(r^2 + W^2)

Since x(t) is the sum of

  A_k*cos(k*t + q_k)

over all k, it follows that y(t) is the sum of

  B_k*cos(k*t + q_k - arctan(k/r)) ,

where

  B_k = (A_k)/sqrt(r^2 + k^2)

Part 3

(3a)

H(s) = s/(4 - s^2) has poles at s = 2, s = -2.

ROC(H) for a BIBO stable system must include s = 0, therefore

  ROC(H) : -2 < Real{s} < 2

in this case. The system is noncausal (ROC is not right-sided.)

(3b)

If x'(t) = u(t), then X(s) = 1/s with ROC(X) : 0 < Real{s} <= infty

Then

  Y'(s) = X'(s)*H(s) = 1/(4 - s^2) = (1/4)/(s+2) - (1/4)/(s-2)

with ROC(Y') = ROC(H) (note pole-zero cancelation).

Thus

  y'(t) = (1/4)*[exp(-2*t)*u(t) + exp(2*t)*u(-t)]

        = exp(-2*|t|)/4   (OK to modify y'(t) at t = 0)

By time invariance, the response to x(t) = x'(t-1) = u(t-1) is

  y(t) = exp(-2*|t-1|)/4 , all t