1. Let X and Y be independently and identically distributed as $X, Y \sim U[-1, 1]$. Let $Z = |X + Y|$. **(7 total points)**

   (a) Find the distribution $F_Z[z]$, for all $z \in \mathbb{R}$. **(3 points)**
   Solution: Given $0 \leq z \leq 2$, considering the triangle on which $X+Y > z$, we find that $\mathrm{P}[X + Y > z] = \frac{1}{4} \cdot \frac{1}{2} \cdot (2 - z)^2 = \frac{1}{8} \cdot (2 - z)^2$. Similarly, $\mathrm{P}[X + Y < -z] = \frac{1}{8} \cdot (2 - z)^2$. Then,

   $$F_Z[z] \;=\; \mathrm{P}[\,|X + Y| \leq z\,] \;=\; \begin{cases} 0 & \text{if } z < 0; \\ 1 - \frac{1}{4} \cdot (2 - z)^2 & \text{if } 0 \leq z < 2; \\ 1 & \text{if } z \geq 2. \end{cases}$$

   (b) Find the expectation $\mathrm{E}[Z]$. **(4 points)**
   Solution: We can first find the density:

   $$f_Z[z] \;=\; \frac{d}{dz} F_Z[z] \;=\; \begin{cases} \frac{1}{2} \cdot (2 - z) & \text{if } 0 \leq z < 2; \\ 0 & \text{otherwise.} \end{cases}$$

   Then,

   $$\mathrm{E}[Z] \;=\; \frac{1}{2} \int_0^2 (2z - z^2)\, dz \;=\; \frac{1}{2}(z^2 - \frac{z^3}{3})\Big|_{z=0}^{2} \;=\; \frac{2}{3}.$$

2. Suppose that $X = 1$ with probability $\frac{1}{2}$, $X = 2$ with probability $\frac{1}{3}$, and $X = 3$ with probability $\frac{1}{6}$. If $X = 1$, then $Y_i \sim U[-2, 2]$, if $X = 2$, then $Y_i \sim U[-1, 1]$, and if $X = 3$, then $Y_i \sim \exp(2)$, for $i = 1, 2, \ldots$, where the Y_i are independent given X. **(7 total points)**

   (a) Given that $Y_1 = -0.8$, find the probability that $X = 3$. **(1 point)**
   Solution:

   $$\mathrm{P}[\,X = 3 | Y_1 = -0.8\,] \;=\; 0.$$

   (b) Given that $Y_1 = -0.8$, find the probability that $X = 2$. **(1 point)**
   Solution: We can first work out the joint densities

   $$f_{X,Y_1}(1, -0.8) \;=\; \mathrm{P}[\,X = 1\,] \cdot f_{Y_1|X}(-0.8|1) \;=\; \frac{1}{2} \cdot \frac{1}{4} \;=\; \frac{1}{8}$$

   and

   $$f_{X,Y_1}(2, -0.8) \;=\; \mathrm{P}[\,X = 2\,] \cdot f_{Y_1|X}(-0.8|2) \;=\; \frac{1}{3} \cdot \frac{1}{2} \;=\; \frac{1}{6}.$$

   The density of the observation is

   $$f_{Y_1}(-0.8) \;=\; f_{X,Y_1}(1, -0.8) + f_{X,Y_1}(2, -0.8) \;=\; \frac{1}{8} + \frac{1}{6} \;=\; \frac{7}{24}$$

   and then

   $$\mathrm{P}[\,X = 2 | Y_1 = -0.8\,] \;=\; \frac{f_{X,Y_1}(2, -0.8)}{f_{Y_1}(-0.8)} \;=\; \frac{\frac{1}{6}}{\frac{7}{24}} \;=\; \frac{4}{7}.$$

(c) Given that $Y_1 = -0.8$ and $Y_2 = 0.5$, find the probability that $X = 2$. **(3 points)**
Solution:

$$\begin{aligned} f_{X,Y_1,Y_2}(1,-0.8,0.5) &= \mathrm{P}[X=1] \cdot f_{Y_1|X}(-0.8|1) \cdot f_{Y_2|X}(0.5|1) = \frac{1}{2} \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{32} \\ f_{X,Y_1,Y_2}(2,-0.8,0.5) &= \mathrm{P}[X=2] \cdot f_{Y_1|X}(-0.8|2) \cdot f_{Y_2|X}(0.5|2) = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{12} \\ f_{X,Y_1,Y_2}(3,-0.8,0.5) &= 0 \end{aligned}$$

so the density of the observation is

$$f_{Y_1,Y_2}(-0.8,0.5) = \frac{11}{96}$$

and then

$$\mathrm{P}[X=2|Y_1=-0.8, Y_2=0.5] = \frac{f_{X,Y_1,Y_2}(2,-0.8,0.5)}{f_{Y_1,Y_2}(-0.8,0.5)} = \frac{\frac{1}{12}}{\frac{11}{96}} = \frac{8}{11}.$$

(d) Given that $Y_1 > 0$, find the (conditional) expectation of Y_1. **(2 points)**
Solution: We can further condition on X:

$$\mathrm{E}[Y_1|Y_1>0, X=1] = 1, \qquad \mathrm{E}[Y_1|Y_1>0, X=2] = \frac{1}{2}, \qquad \mathrm{E}[Y_1|Y_1>0, X=3] = \frac{1}{2}$$

and

$$\mathrm{P}[X=1|Y_1>0] = \frac{3}{7}, \qquad \mathrm{P}[X=2|Y_1>0] = \frac{2}{7}, \qquad \mathrm{P}[X=3|Y_1>0] = \frac{2}{7},$$

and then

$$\mathrm{E}[Y_1|Y_1>0] = \frac{3}{7} \cdot 1 + \frac{2}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{1}{2} = \frac{5}{7}$$

3. Let X be chosen uniformly from the set $\{1, \ldots, n\}$. Then let Y be chosen uniformly from the set $\{1, \ldots, X\}$. You may assume that $n \geq 5$. **(6 total points)**

   (a) Find the probability $\mathrm{P}[X+Y=4]$. **(2 points)**
   Solution:

$$\begin{aligned}
\mathrm{P}[X+Y=4] &= \mathrm{P}[X=2]\,\mathrm{P}[Y=2|X=2] + \mathrm{P}[X=3]\,\mathrm{P}[Y=1|X=3] \\
&= \frac{1}{n}\cdot\frac{1}{2} + \frac{1}{n}\cdot\frac{1}{3} \;=\; \frac{5}{6n}.
\end{aligned}$$

   (b) Find the expectation $\mathrm{E}[Y]$. **(4 points)**
   Solution: For $k \in \{1, \ldots, n\}$,

$$\mathrm{P}[Y=k] = \sum_{j=k}^{n} \mathrm{P}[X=j]\,\mathrm{P}[Y=k|X=j] = \sum_{j=k}^{n} \frac{1}{n}\cdot\frac{1}{j}$$

   Then,

$$\begin{aligned}
\mathrm{E}[Y] &= \sum_{k=1}^{n} k\cdot\mathrm{P}[Y=k] = \frac{1}{n}\sum_{k=1}^{n} k \sum_{j=k}^{n}\frac{1}{j} = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=k}^{n}\frac{k}{j} \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{j}\frac{k}{j} = \frac{1}{n}\sum_{j=1}^{n}\frac{j+1}{2} \\
&= \frac{1}{2n}\cdot\frac{n(n+3)}{2} = \frac{n}{4}+\frac{3}{4}.
\end{aligned}$$