Qualifying Examination Basic Mathematics Fall 2017

(1) Calculate the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ (2 pts.)

What are the eigenvalues of $\mathbf{A}$? (3 pts.) Find one of the normalized eigenvectors. (1 pt.)

Hint: one eigenvalue is 6.

(2) Solve in terms of an infinite series the differential equation

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$, subject to the boundary condition $y(0) = 1$ (7 pts.)

(3) What are the equations of the two straight lines that are tangent to the circle $(x-1)^2 + (y-2)^2 = 4$ at the points on the circle where $x = 2$. (4 pts.) What is the angle between these two straight lines? (3 pts.)

Solutions

(1) Inverse is $\begin{pmatrix} 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \\ -5/18 & 1/18 & 7/18 \end{pmatrix}$

The characteristic is equation $\lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$. Given that one eigenvalue is 6 this can be factored as $(\lambda - 6)(\lambda^2 - 3) = 0$. The other two eigenvalues are $\pm\sqrt{3}$. For the eigenvalue of 6 the normalized eigenvector is $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

(2) The method of Frobenius suggest a solution of the form $y = e^b\left(a_0 + a_1x + a_2x + a_3x^2 + a_4x^3 + ...\right)$. If this is substituted in the differential equation it is easy to find that b=0. Substituting $y = \left(a_0 + a_1x + a_2x + a_3x^2 + a_4x^3 + ...\right)$ in the equation and ensuring that the coefficient of $x^n = 0$ gives the solution as

$$y = 1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 \times 4^2}x^4 - \frac{1}{2^2 \times 4^2 \times 6^2}x^6 + ...$$

(3) The two tangent points are $(2, 2+\sqrt{3}), (2, 2-\sqrt{3})$. The unit tangent vector at the first point is $\left(\frac{\mathbf{i} - \frac{\mathbf{j}}{\sqrt{3}}}{\sqrt{\frac{4}{3}}}\right)$. The unit vector at the second point is $\left(\frac{\mathbf{i} + \frac{\mathbf{j}}{\sqrt{3}}}{\sqrt{\frac{4}{3}}}\right)$. These are found by setting the dot product of radius vector and tangent vector =0. The dot product of these two unit vectors give the cosine of the angle between them. This gives $\theta = \arccos(1/2) = 60^o$.

The slopes of the two tangent lines are $\frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$. The first tangent equation is $y = \frac{-x}{\sqrt{3}} + \frac{5 + 2\sqrt{3}}{\sqrt{3}}$.

The second tangent equation is $y = \frac{x}{\sqrt{3}} + \frac{2\sqrt{3} - 5}{\sqrt{3}}$