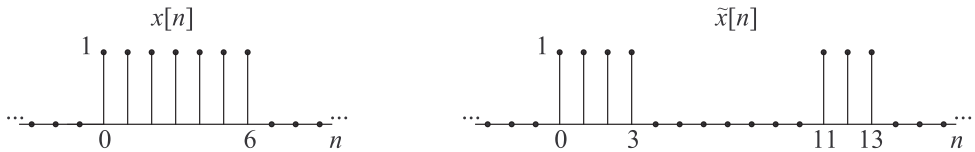
Linear Systems and Signals — Ph.D. Qualifying Exam, August 2017

Part 1 (6 pts.)

Sequences $x[n]$ and $\tilde{x}[n]$ both have finite duration, with nonzero samples as shown below.



(1a) (3 pts.) Give an example of a sequence $v[n]$ consisting entirely of nonzero samples (i.e., $v[n] \neq 0$ for all n) such that the convolution of $v[n]$ and $x[n]$ (left graph) is the all-zeros sequence:

$$(\forall n \in \mathbb{Z}) \qquad (v * x)[n] \ = \ 0$$

(1b) (3 pts.) Suppose the response of a linear time-invariant system to the input sequence $x[n]$ is the (output) sequence $y[n]$. What is the response $\tilde{y}[n]$ of the same system to the input sequence $\tilde{x}[n]$ (right graph)? Express $\tilde{y}[\,\cdot\,]$ in terms of $y[\,\cdot\,]$.

Part 2 (7 pts.)

Consider the *periodic* signal $x(t)$ of period $T_0 = 6$ sec, defined (over the time interval $[-3, 3]$) by

$$x(t) \ = \ \begin{cases} 1, & |t| \leq 1 \\ 0, & 1 < |t| \leq 3 \end{cases}$$

(2a) (3 pts.) Sketch the general form of the Fourier transform $X(j\omega)$, where ω is in rad/sec. The horizontal (ω) axis should be marked in detail. You need not provide values for the vertical coordinates.

(2b) (4 pts.) If $x(t)$ is the input to a linear time-invariant system with frequency response

$$H(j\omega) \ = \ \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & |\omega| > \pi/2 \end{cases},$$

determine the system's output $y(t)$.

Part 3 (7 pts.)

Consider the *causal* linear time-invariant system with transfer function

$$H(s) \ = \ \frac{s-1}{s^2 + bs + 3} \ ,$$

where b is real-valued.

(3a) (4 pts.) For which, if any, values of b does the system's impulse response $h(t)$ have *both* of the following properties:

- it is bounded (over all time t); *and*
- there are infinitely many values of t greater than zero for which $h(t) = 0$.

(3b) (3 pts.) If $b = 1$, determine the response (output) $y(t)$ of the system to the input signal given by $x(t) = \cos t$, for all $t \in \mathbb{R}$. Your answer should be real-valued.

Part 1

(1a)

If z[.] is the convolution of x[.] and v[.], then

    z[n] = v[n] + ... + v[n-6]

This can be made zero for all n by choosing v[.]
periodic with period 7 samples, and sum = 0 over one
period, e.g.,

    v[0 through 6] = (1, 2, 3, 4, 5, 6, -21)

(1b)

(Using ' instead of ~)

    x'[n] = x[n] - x[n-4] + x[n-7]

By linearity and time invariance,

    y'[n] = y[n] - y[n-4] + y[n-7]

Part 2

Period T0 = 6 sec => Fundamental w0 = 2*pi/6 = pi/3 rad/sec

    x(t) = sum of X[k]*exp(j*k*w0*t) over all integers k

(i.e., X[k] is the kth Fourier series coefficient)

(2a)

X(j*w) = sum of 2*pi*X[k]*delta(w-k*w0) over all k

Thus the graph consists exclusively of impulses at positions

    w = k*W0 = k*(pi/3)

(2b)

H(j*w) is an ideal lowpass filter characteristic with
cutoff frequency wc = pi/2 and gain = 1. Harmonics k = -1, 0
and +1 are within the passband |w|<|wc|.

    X[0] = (1/6)*(area under x(.) over [-3,3]) = 1/3

    X[1] = (1/6)*(integral of exp(-j*pi*t) over [-3,3])

         = sin(pi/3)/pi = sqrt(3)/(2*pi)

Thus y(t) = 1/3 + (sqrt(3)/pi)*cos(pi*t/3).

Part 3

(3a)

h(t) = [C1*exp(a1*t) + C2*exp(a2*t)]*u(t)

where a1 and a2 are the poles (roots of the quadratic denominator of H(s)).

h(.) is bounded iff the real parts of a1 and a2 are both =< 0; and it has infinitely many zero crossings in positive time iff the imaginary parts of a1 and a2 are nonzero. Since

  a1 = (-b + sqrt(b^2 - 12))/2 , (-b - sqrt(b^2 - 12))/2 ,

the conditions are met iff

  b >= 0 and b^2 - 12 < 0, i.e., b is in [0, 2*sqrt(3))

(3b)

If x(t) = cos(t) for all t, then

   y(t) = A*cos(t + q) ,

where A*exp(j*q) = H(j*w) at w = 1. Since

  H(j*1) = (j - 1)/(-1 + j + 3) = -(1/5) + j*(3/5)

         = (sqrt(10)/5)*exp(j*(-arctan(3)+pi))

it follows that

  y(t) = (sqrt(10)/5)*cos(t-arctan(3)+pi)

  (also) = -(1/5)*cos(t) - (3/5)*sin(t)