Qual exam ECE/ Basic Physics: 2017

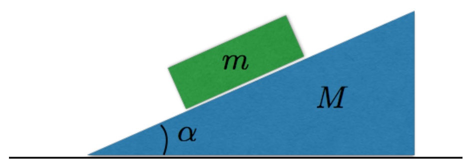
1. Sliding blocks (6 pts)

A block of mass m is located on a wedge of mass M and angle α. The wedge is positioned on a surface. We ignore friction in this problem.

(a) Show all forces on both blocks, i.e., the force diagram. (2 pts)

(b) a_1 is the acceleration of m with respect to M, and a_2 is the acceleration of M with respect to the surface. Indicate a_1 and a_2 on the diagram. Using the force diagram from part (a) find the relation between acceleration and force, i.e., equations of motions. (2pts)

(c) Find a_1 and a_2. (2pts)

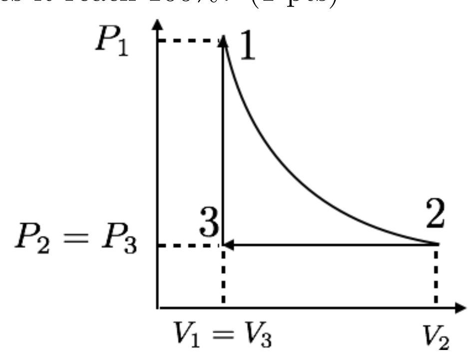


2. Ideal gas engine (7 pts)

Suppose an ideal gas is subjected to the cyclic process shown in the figure, with temperature T_1,T_2 and T_3 in states 1, 2 and 3, respectively. Recall that for an ideal gas $PV = nRT$, where n is the number of moles and R is the universal gas constant. $1 \rightarrow 2$ is an isothermal expansion, $2 \rightarrow 3$ is an isobaric expansion, and $3 \rightarrow 1$ is an isochoric heating step. All steps are reversible

(a) Find the heat (ΔQ), the work done on the gas (ΔW) and the change in the internal energy (ΔU) for each step. (3 pts)
(b) What's the total ΔQ, ΔW and ΔU, for the overall cycle? (3 pts)

(c) What is the efficiency, the ratio between total work and the input heat, of this cycle? In which limit does it reach 100%? (1 pts)



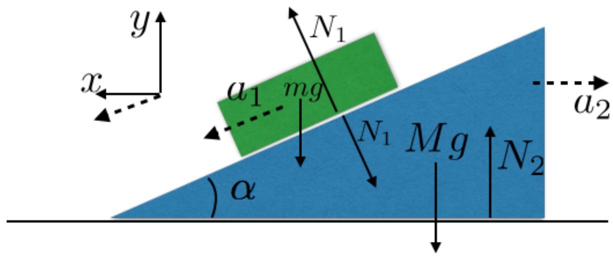
3. Quantum Harmonic Oscillator and LC-circuit (7 pts)

(a) Write down the Hamiltonian for a harmonic oscillator with the mass m and the angular frequency ω. According to Virial's theorem, the expectation value of the kinetic and potential energy should be equal to each other for an harmonic oscillator. Explicitly write down this statement in terms of position $\hat{x}$ and momentum $\hat{p}$ operators and other parameters in the system. (2 pts)

(b) Which eigen-energy state has the smallest <u>absolute mean value</u> $|\langle \hat{x} \rangle|$ and <u>variance</u> $\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$, both in position and momentum? Using the Heisenberg's uncertainty principle and the result from the previous part, find the standard deviation in position associated with that state, i.e., the zero point fluctuation of position $x_{\text{zpf}} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$. (3 pts)

(c) Now consider an LC-circuit, where the kinetic energy comes from the inductor, and the potential energy is stored in the capacitor $H = \frac{L}{2}(\frac{dQ}{dt})^2 + \frac{Q^2}{2C}$. In this analogy, the inductor plays the role of mass $L \to m$, and also $\frac{1}{C} \to m\omega^2$, and naturally $\frac{1}{LC} \to \omega$. In other words, one can treat the charge (Q) in a quantum manner, similar to the position of a harmonic oscillator ($\hat{x}$). Given the above relations and the result of the previous part, what's the smallest fluctuation in charge in an LC circuit, in terms of the characteristic impedance $Z = \sqrt{\frac{L}{C}}$? (2 pts)

1. Solution: sliding block



a_1 denotes the acceleration of m w.r.t. M and a_2 denotes the acceleration of M w.r.t. ground. The acceleration of m w.r.t. is given by vector sum of a_1 and a_2.

The equation of motion for m, along the x-direction is: $N_1 \sin\alpha = m(a_1 \cos\alpha - a_2)$ along the incline $mg \sin\alpha = m(a_1 - a_2 \cos\alpha)$. The equation of motion for M along the x-direction $-N_1 \sin\alpha = -Ma_2$.

Combing the first and the third equations, we have: $a_1 = \frac{M+m}{m\cos\alpha} a_2$. If we plug this back into the second equation of motion, we get: $a_2 = g\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + \frac{M}{m}}$.

2. Solution: Ideal gas

* $1 \to 2$ Isothermal $dT = 0 \to dU = 0$ $\left(dU = dQ - PdV = dQ + dW \right)$

$$\Delta Q_{12} = -\Delta W_{12} = + nRT_1 \ln(V_2/V_1)$$

* $2 \to 3$ Isobaric

$$\Delta Q_{23} = nC_p \Delta T = nC_p (T_3 - T_2) < 0$$

$$\Delta W_{23} = -\int_{V_2}^{V_3} PdV = P_2 (V_2 - V_3) = \frac{nRT_2}{V_2}(V_2 - V_3).$$

since $V_3 = V_1$, $T_2 = T_1$ then $\Delta W_{23} = nRT_1 \left(1 - \frac{V_1}{V_2}\right)$

$$\Delta U_{23} = \Delta Q_{23} + \Delta W_{23}$$

* $3 \to 1$ no change in volume $\to \Delta W_{31} = 0$

$$\Delta Q_{31} = C_V (T_1 - T_3)$$

$$\Delta U_{31} = \Delta Q_{31}$$

* total ΔU is zero, since we have an ideal gas.

$$\Delta W_{total} = nRT_1 \left[1 - \frac{V_1}{V_2} - \ln\frac{V_2}{V_1} \right] < 0$$ $\to$ the system is doing work.

Efficiency $\eta = \frac{W_{tot}}{Q_{in}} = \frac{W_{tot}}{\Delta Q_{12}} = 1 - \frac{1 - V_2/V_1}{\ln V_2/V_1}$, if $\frac{V_1}{V_2} \to 0$ then $\eta \to 1$

3. Solution: Quantum Harmonic Oscillator and LC-circuit

(a) $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. Virial's theorem gives: $\frac{\langle\hat{p}^2\rangle}{2m} = \frac{1}{2}m\omega^2\langle\hat{x}^2\rangle$.

(b) The ground state with energy $\frac{1}{2}\hbar\omega = \frac{\langle\hat{p}^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle\hat{x}\rangle^2$ has the smallest position and momentum variance and zero mean values: $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$. Heisenberg's uncertainty principle for the ground state gives $\sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} \times \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} = \hbar/2$. For the ground state, this reduces to $\langle\hat{p}^2\rangle = \frac{\hbar}{2\langle\hat{x}^2\rangle}$. Combined with the result above, we have $x_{\text{zpf}}^2 = \langle\hat{x}^2\rangle = \frac{\hbar}{2m\omega}$, or $x_{\text{zpf}} = \sqrt{\frac{\hbar}{2m\omega}}$.

(c) Again for the ground state, knowing $\frac{1}{\sqrt{LC}} \to \omega$ and $L \to m$, for the charge variance, we get $Q_{\text{zpf}} = \sqrt{\frac{\hbar}{2}\sqrt{\frac{C}{L}}} = \sqrt{\frac{\hbar}{2Z}}$.