PROBABILITY - Ph.D. Qualifying Exam Spring 2017

Part 1: Consider that two random variables X and Y are uniformly distributed in the set $\mathbb{A}$ depicted in Figure 1.

A) **(2 pt)** Determine f_{XY}, the joint PDF (probability density function) of X and Y.

B) **(2 pt)** Determine f_Y, the PDF of Y.

C) **(3 pt)** Compute the mean and variance of Y.

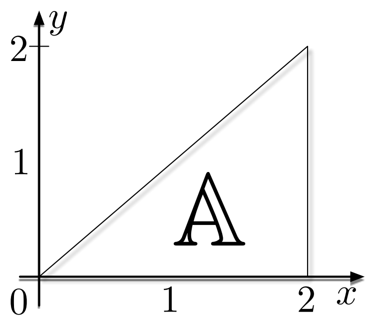

Figure 1: The joint PDF $f_{X,Y}$ is uniformly distributed in $\mathbb{A}$ and is 0 outside of $\mathbb{A}$.

Solution:
Part A: `The joint PDF is` 0.5 `inside` $\mathbb{A}$ `and zero outside.`
Part B: $f_Y(y) = 0.5 * (2 - y)$ `for` $y \in [0, 2]$ `and is zero outside.`
Part C: `Mean is` $2/3$, $E[Y^2] = 2/3$, `variance is` $2/3 - (2/3)^2 = 2/9$

Part 2: Let X_1 and X_2 be independent random variables exponentially distributed with expected value 1. Compute the CDF (cumulative distribution function) and the PDF (Probability density function) for the following random variables:

A) **(3 pts)** $Z = \max\{X_1, X_2\}$

B) **(3 pts)** $Z = \frac{X_1}{X_2+1}$

Solution
Part A: The CDF of Z is

$$F_Z(z) = \begin{cases} (1-e^{-z})^2 & z \geq 0 \\ 0 & z < 0 \end{cases}$$

and PDF is

$$f_Z(z) = \begin{cases} 2e^{-z}(1-e^{-z}) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Part B: The CDF is

$$F_Z(z) = \begin{cases} 1 - \frac{e^{-z}}{z+1} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

and the PDF is

$$f_Z(z) = \begin{cases} \frac{z+2}{(z+1)^2}e^{-z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Part 3: A transmission system has three components: (1) an encoder(transmitter), (2) a channel and (3) a decoder(receiver). There are two possible symbols $\{M_1, M_2\}$, out of which one is selected randomly. Each symbol is equally likely to be selected. The encoder uses a codebook that assigns to each symbol a string of N bits (we assume N is odd). To M_1 it assigns a bit string of all zeroes and to M_2 it assigns a bit string of all ones. The channel transmits a bit string from the encoder to the decoder, and in the process it will flip each bit (independently) with probability $p = 0.25$. The decoder will use the received bit string to attempt to resolve which symbol was selected. Its output is $\hat{M}$, which is M_1 if the received bit string has more zeroes than ones, and otherwise it declares M_2.

A) (4 %) Calculate the exact probability of error (i.e., $\hat{M}$ is not the selected symbol) for $N = 5$.

B) (3 %) What would be the probability of error as N tends to infinity? (Give a precise and short proof of your answer)

Solution:
Part A) Using Binomial $P(Error) = \frac{106}{1024}$
Part B) Use Chebyshev's inequality to show that it tends to zero.