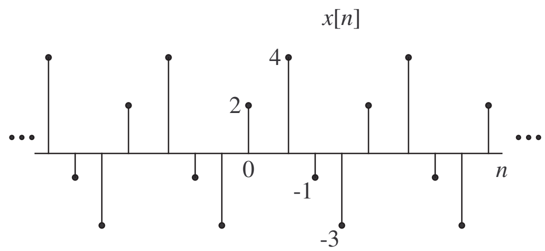
Linear Systems and Signals — Ph.D. Qualifying Exam, January 2017

Part 1 (3 pts.)

The sequence $x[n]$ shown below is periodic with period equal to four samples.



If $x[n]$ is the input to a filter with input-output relationship

$$y[n] \;=\; x[n] \;-\; 2x[n-1] \;+\; 3x[n-2] \;-\; 2x[n-3] \;+\; x[n-4] \;,$$

determine the output sample $y[30]$.

Part 2 (4 pts.)

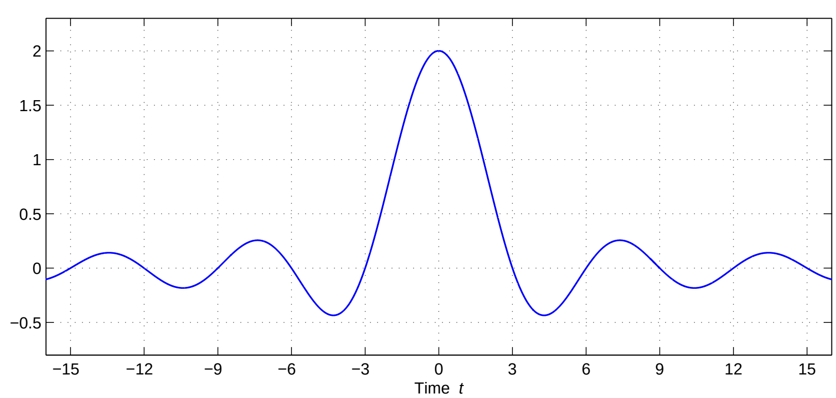
A linear system operating in continuous time has input-output relationship

$$y(t) \;=\; x(t-1) \;+\; e^{-2t} \cdot \int_{t-2}^{\infty} x(\tau) e^{2\tau} \, d\tau$$

Is the system causal? Is it time invariant? *Explain.* If it is time-invariant, give an equation for its impulse response $h(t)$.

Part 3 (4 pts.)

The impulse response $h(t)$ of an *ideal* lowpass filter is plotted below over a time interval which is symmetric about the origin $t = 0$.



Determine and *sketch* the filter's frequency response $H(j\omega)$.

Part 4 (4 pts.)

Let $X(s)$ be the Laplace transform of

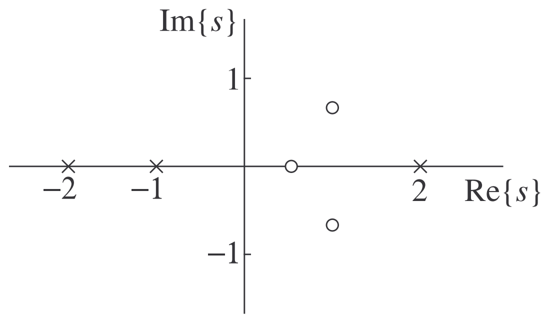
$$x(t) \;=\; t^{4/3} \cdot 4^{-t} u(t) \;+\; \frac{3^t}{1+|t|} u(1-t) \;,$$

where $u(t)$ is the unit step function at $t=0$.

What is the region of convergence of $X(s)$? (*Do not attempt to determine* $X(s)$ *itself.*)

Part 5 (5 pts.)

Shown below is the zero-pole plot for the transfer function $H(s)$ of a linear time-invariant system operating in continuous time. As usual, "$\circ$" denotes a zero and "$\times$" denotes a pole. Note that the exact positions of the zeros are *not* given.



Assuming the system in question is *BIBO stable*, determine the *general* form of its impulse response $h(t)$ (using coefficients of unspecified value).

LINEAR SYSTEMS: SOLUTIONS

Part 1
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The output has the same period as the input (4 samples)

y[30] = y[2] = x[2] - 2*x[1] + 3*x[0] - 2*x[-1] + x[-2]

= -1 - 2*4 + 3*2 - 2*(-3) + (-1) = 2

Part 2
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The system is not causal, since y(t) depends on values of the input
x(.) over the time interval [t,infty).

The expression for y(t) in terms of the input is given by

y(t) = x(t-1)

+ int{x(tau)*exp(2*(t-tau))*d(tau) over tau in [t-2,infty)},

and the integral can be further written as

int{x(tau)*exp(2*(t-tau))*u(tau-t+2)*d(tau) over tau in [-infty,infty)}

Therefore y(t) is the convolution of x(t) and

h(t) = delta(t-1) + exp(2*t)*u(2-t) ,

which also implies that the system is time-invariant.

Part 3
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The ideal lowpass filter characteristic is

H(j*w) = A for |w|<|w0| , 0 for |w|>|w0|

(Since h(t) is even-symmetric, there is no exp(-j*w*delay) factor in
H(j*w).)

Therefore

h(t) = (A/(2*pi))*integral{H(j*w)*exp(j*w*t)*dw over (-infty,infty)}

= (A/pi)*sin(w0*t)/t

h(t) = 0 at t = (nonzero integer)*pi/w0

=> (from graph) 3 = pi/w0 => w0 = pi/3

h(0) = A*w0/pi => (from graph) 2 = A*w0/pi

=> A = 6

Part 4

x(t) is the sum of a left-sided and a right-sided exponential, each multiplied by a subexponential factor whose integral (over the corresponding half-line) diverges. ROC(X) will therefore be the intersection of the ROC's of the two exponentials. Noting that

4^(-t) = exp(a*t) for a = -ln(4) and 3^t = exp(b*t) for b = ln(3),

we obtain

ROC(X) : -2*ln(2) < Real{s} < ln(3)

Part 5

The ROC of a stable system must include the imaginary axis; equivalently, all exponentials in h(.) should decay in the appropriate direction. In this case,

ROC(H) : -1 < Re{s} < 2

Since neither a reference gain nor the positions of the zeros are given, the coefficients of the partial fraction expansion cannot be specified. Thus h(t) will be a linear combination of:

- exp(-2*t)*u(t) ;

- exp(-t)*u(t) ;

- exp(2*t)*u(-t) ; and

- delta(t) (we have an equal number of zeros and poles on the finite s-plane, therefore the partial-fraction expansion of H(s) will include a constant term.)