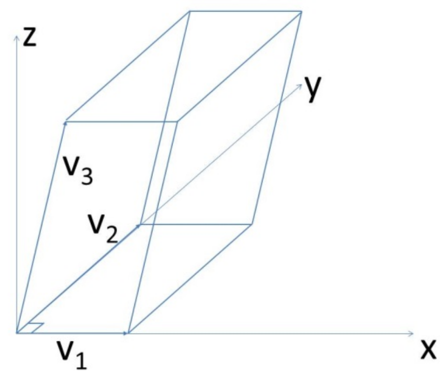
Qualifying Examination Basic Mathematics 2017

(1) (6 pts.)A Parallelepiped has one corner at the point (0,0,0) in Cartesian coordinates. The vectors from this corner that delineate the three sides of the parallelepiped at this point are:



$\mathbf{v}_1 = 2\hat{\mathbf{i}}$

$\mathbf{v}_2 = 3\hat{\mathbf{j}}$

$\mathbf{v}_3 = \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors in the x,y, and z directions, respectively.

(a) What is the volume of the parallelepiped?
(b) What is the total surface area of the parallelepiped?
(c) What is the outward pointing unit vector from each of the two tilted faces of the parallelepiped

(2) (7 pts.)What is the solution to the differential equation

$2\frac{dy}{dt} + 3y = e^{-t}\cos(5t)$ subject to the boundary condition $y(0) = 1$

(3) (7 pts.) A matrix transformation of the form $\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ where all the matrices are square, non-singular matrices is called a similarity transformation. Prove that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues. Prove also that $\mathrm{Tr}\left(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\right) = \mathrm{Tr}(\mathbf{A})$, where Tr indicates the trace of the matrix (the sum of its diagonal elements). What are the eigenvalues and normalized eigenvectors of the matrix $\begin{pmatrix} 0 & 2 \\ -4 & -6 \end{pmatrix}$?

Solutions

(1) $|\mathbf{v}_1| = 2$ $|v_2| = 3$ $|v_3| = \sqrt{17}$

$\mathbf{v}_1 = 2\hat{\mathbf{i}}$, $\mathbf{v}_2 = 3\hat{\mathbf{j}}$, $\mathbf{v}_3 = \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

Note that $\mathbf{v}_1 \bullet \mathbf{v}_2 = 0$, $\mathbf{v}_1 \bullet \mathbf{v}_3 = 0$

Volume is $\mathbf{v}_1 \bullet (\mathbf{v}_2 \times \mathbf{v}_3) = 2\hat{\mathbf{i}}12\hat{\mathbf{i}} = 24$

The area of a rhomboid bounded by vectors **a** and **b** is $area = |\mathbf{a} \times \mathbf{b}|$

area of rectangular end faces = 6 each

area of rectangular side faces= $2\sqrt{17}$

area of remaining two faces is 12 each

Total surface area is 36+4 $\sqrt{17}$

One output pointing vector is $\mathbf{v}_3 \times \mathbf{v}_1 = 8\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$. Unit vector is $\dfrac{8\hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{68}}$

Second output pointing unit vector is $-8\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Unit vector is $\dfrac{-8\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{68}}$

(2) The equation can be re-written as

$$\frac{dy}{dt} + 3y/2 = e^{-t}\cos(5t)/2 = \tfrac{1}{4}e^{-t}\left(e^{j5t} + e^{-j5t}\right)$$

For homogeneous equation integration factor is $e^{\frac{3}{2}t}$ which gives

$$ye^{\frac{3}{2}t} = \tfrac{1}{2}\int e^{\frac{t}{2}t}\left(\cos(5t)\right)dt = \tfrac{1}{4}\int\left(e^{\frac{1}{2}t}\left(e^{j5t} + e^{-j5t}\right)\right)dt = \tfrac{1}{4}\left(\frac{e^{\frac{1}{2}t+j5t}}{0.5+j5} + \frac{e^{\frac{1}{2}t-j5t}}{0.5-j5}\right) + A.$$

where *A* is a constant

$$y = \tfrac{1}{4}\left(e^{-t+j5t}\,\frac{(0.5-j5)}{25.25} + e^{-t-j5t}\,\frac{(0.5+j5)}{25.25}\right) + Ae^{-\frac{3}{2}t} = \tfrac{1}{101}e^{-t}\left(e^{j5t}(0.5-j5) + e^{-j5t}(0.5+j5)\right) + Ae^{-\frac{3}{2}t}$$

which gives

$y = \frac{1}{101} e^{-t} \left(\cos(5t) - 10\sin(5t)\right) + A^{-\frac{3}{2}t}$, which from boundary condition gives *A* =-1/101.

(3) If $\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ then $\lambda\mathbf{I} - \mathbf{B} = \lambda\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{Q}^{-1}[\lambda\mathbf{I} - \mathbf{A}]\mathbf{Q}$, where $\mathbf{I}$ is an identity matrix

Moreover $|\lambda\mathbf{I} - \mathbf{B}| = |Q^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{Q}| = |\lambda\mathbf{I} - \mathbf{A}|$ so the two matrices related by a similarity equation have the same characteristics equations and eigenvalues.

$$Tr(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}) = Tr\left(\mathbf{A}\mathbf{Q}\mathbf{Q}^{-1}\right) = Tr(\mathbf{A})$$

Characteristic equation is $\lambda^2 + 6\lambda + 8 = 0$, which gives the eigenvalues as -4, and -2.

The corresponding normalized eigenvectors are $\begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$ and $\begin{pmatrix} -0.447 \\ 0.894 \end{pmatrix}$