1. Suppose that, to get to work, one can either drive or take the train. When driving, the amount of time (in minutes) it takes is uniformly distributed over the interval $[15, 25]$. When taking the train, the time waiting for the train is exponentially distributed with parameter $1/5$ (that is, with pdf of $\frac{1}{5}e^{-w/5}$ for $w \geq 0$ and 0 otherwise), followed by the train ride which will take 15 minutes (such that the mean overall travel time is the same for each). **(6 total points)**

   (a) Suppose that this person has an important meeting in 18 minutes, and find the probability of being late with each of the two options.

   (3 points)

   Solution: Let D be the time it takes to drive, W the time waiting for the train, and T the total time with the train.

$$\mathrm{P}[\,D > 18\,] = \frac{25-18}{25-15} = \frac{7}{10} \qquad\qquad \mathrm{P}[\,T > 18\,] = \mathrm{P}[\,W > 3\,] = e^{-3/5}$$

   (b) Find the probability that taking the train would take longer than driving. **(3 points)**

   Solution:

$$\begin{aligned}
\mathrm{P}[\,T > D\,] &= \int_{15}^{25} \mathrm{P}[\,T > r\,] \cdot \frac{1}{10}\, dr \\
&= \int_{15}^{25} \mathrm{P}[\,W > r - 15\,] \cdot \frac{1}{10}\, dr \\
&= \frac{1}{10} \cdot \int_{15}^{25} e^{-\frac{r-15}{5}}\, dr \\
&= -\frac{5}{10} e^{-\frac{r-15}{5}} \Big|_{r=15}^{25} \\
&= \frac{1}{2}\left(1 - e^{-2}\right)
\end{aligned}$$

2. An urn contains 4 white balls and 6 black balls. A (first) ball is chosen at random. It is then replaced, along with 2 more balls of the same color (such that there are then 12 balls in the urn). **(7 total points)**

   (a) Another (second) ball is then drawn at random from the urn. Find the probability that this ball is white. **(2 points)**

   Solution:

$$\begin{aligned} \mathrm{P}[B_2 = w] &= \mathrm{P}[B_1 = w]\,\mathrm{P}[B_2 = w \mid B_1 = w] + \mathrm{P}[B_1 = b]\,\mathrm{P}[B_2 = w \mid B_1 = b] \\ &= \frac{2}{5}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{1}{3} \\ &= \frac{2}{5} \end{aligned}$$

   (b) Given that this second ball is white, find the probability that the first ball drawn from the urn was black. **(3 points)**

   Solution:

$$\begin{aligned} \mathrm{P}[B_1 = b \mid B_2 = w] &= \frac{\mathrm{P}[B_1 = b, B_2 = w]}{\mathrm{P}[B_2 = w]} \\ &= \frac{\mathrm{P}[B_1 = b]\,\mathrm{P}[B_2 = w \mid B_1 = b]}{\mathrm{P}[B_2 = w]} \\ &= \frac{\frac{3}{5}\cdot\frac{1}{3}}{\frac{2}{5}} = \frac{1}{2} \end{aligned}$$

   (c) The second ball is replaced (such that there are still 12 balls in the urn), and a third ball is drawn. If the second and third balls are both white, find the probability that the first ball was black. **(2 points)**

   Solution:

$$\begin{aligned} \mathrm{P}[B_2 = w, B_3 = w] &= \mathrm{P}[B_1 = w]\,\mathrm{P}[B_2 = w, B_3 = w \mid B_1 = w] \\ &\quad + \mathrm{P}[B_1 = b]\,\mathrm{P}[B_2 = w, B_3 = w \mid B_1 = b] \\ &= \frac{2}{5}\cdot\left(\frac{1}{2}\right)^2 + \frac{3}{5}\cdot\left(\frac{1}{3}\right)^2 \\ &= \frac{1}{10} + \frac{1}{15} = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \mathrm{P}[B_1 = b \mid B_2 = w, B_3 = w] &= \frac{\mathrm{P}[B_1 = b, B_2 = w, B_3 = w]}{\mathrm{P}[B_2 = w, B_3 = w]} \\ &= \frac{\mathrm{P}[B_1 = b]\cdot\mathrm{P}[B_2 = w, B_3 = w \mid B_1 = b]}{\mathrm{P}[B_2 = w, B_3 = w]} \\ &= \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{2}{5} \end{aligned}$$

3. Suppose that a point Z is picked uniformly at random from the perimeter of a unit circle; that is, from a circle of radius 1 with center at the origin, $(0,0)$. Now let X be the x-coordinate of this point Z. **(7 total points)**

   (a) Find the distribution and density of X. **(5 points)**

   Solution: Given $-1 \leq x \leq 1$,

$$F_X[x] \;=\; \mathrm{P}\,[\,X \leq x\,] \;=\; 2 \cdot \mathrm{P}\left[\,\cos^{-1}(x) \leq \Theta \leq \pi\,\right] \;=\; \frac{\pi - \cos^{-1}(x)}{\pi}$$

   where $\theta = \cos^{-1}(x)$ is given by $\theta \in [0, \pi]$ such that $\cos(\theta) = x$, and so

$$F_X(x) \;=\; \begin{cases} 0 & \text{if } x < -1; \\ \frac{\pi - \cos^{-1}(x)}{\pi} & \text{if } -1 \leq x < 1; \\ 1 & \text{if } x \geq 1. \end{cases}$$

   Then,

$$f_X(x) \;=\; \frac{d}{dx} F_X(x) \;=\; \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{if } -1 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

   Alternatively, one could find the density first in a few ways, and then integrate to get the distribution.

   (b) Let $Y = |X|$. Find the expectation $\mathrm{E}\,[Y]$. **(2 points)**

   Solution:

$$\mathrm{E}\,[Y] \;=\; 2 \cdot \int_0^1 \frac{x}{\pi\sqrt{1-x^2}}\,dx \;=\; -\frac{2}{\pi}\sqrt{1-x^2}\,\Big|_{x=0}^{1} \;=\; \frac{2}{\pi}$$