Fall 2018 Written Qualifying Examination

Basic Mathematics

1. (a) (3 pts) Provide a mathematically rigorous definition for "the sequence $\{x_k\}_{k=1}^{\infty}$ converges to x^*", where $\{x_k\}_{k=1}^{\infty}$ is a scalar sequence, and x^* is a (finite) scalar.

   (b) (4 pts) For $k = 1, 2, \ldots$, let $x_k = \sum_{\ell=1}^{k} \frac{1}{\ell}$. Prove or disprove: $\{x_k\}_{k=1}^{\infty}$ converges to some (finite) x^*. [NOTE: Significant partial credit will be given if key steps are taken.]

2. Let A be a real, symmetric (square) matrix.

   (a) (3 pts) Prove that all eigenvalues of A are real.

   (b) (3 pts) Prove that the eigenvectors of A associated to different eigenvalues are orthogonal.

3. Consider the system of first order differential equations $y'(t) = Ay(t)$, where A is a real square matrix, t is a scalar (time), y lies in R^n, and y' is the derivative of y with respect to t.

   (a) (3 pts) Let $n = 2$, and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (harmonic oscillator). Prove or disprove: There exists a vector $v \neq 0$ such that the solution to the system with auxiliary condition $y(0) = v$ entirely lies on straight line in R^2.

   (b) (4 pts) Let $n = 5$. Prove or disprove: There always exists a vector $v \neq 0$ such that the solution to the given system with auxiliary condition $y(0) = v$ entirely lies on straight line in R^5. [Hint: Odd degree (univariate) polynomial equations with real coefficients have at least one real root.]

Solutions

1. (a) The sequence $\{x_k\}_{k=1}^{\infty}$ converges to x^* if, given $\epsilon > 0$, there exists a positive integer N such that $|x_k - x^*| < \epsilon$ for every $k \geq N$.

   (b) Disproof. For given n, $\sum_{k=n+1}^{2n} \frac{1}{k} > n\frac{1}{2n} = \frac{1}{2}$. Hence, for $k = 2^N$, N any positive integer, $x_k = \sum_{\ell=1}^{k} \frac{1}{\ell} = 1 + \sum_{n=1}^{N} \sum_{\ell=2^{n-1}+1}^{2^n} \frac{1}{\ell} > 1 + \frac{N}{2}$. Since $\{x_k\}$ is monotonically increasing, it follows that $x_k \to \infty$ as $k \to \infty$.

2. (a) Let λ be an eigenvalue of A, and let $v \neq 0$ be an associated eigenvector. Then

$$\lambda^*(v^*)^T v = (v^*)^T A^T v = (v^*)^T A v = (v^*)^T \lambda v = \lambda (v^*)^T v,$$

   so that $(\lambda - \lambda^*)\|v\|^2 = 0$, hence (since $v \neq 0$), $\lambda = \lambda^*$.

   (b) Let $\lambda_1 \neq \lambda_2$ be (real) eigenvalues of A, and v_1, v_2 be associated (real) eigenvectors. Then

$$\lambda_1 v_2^T v_1 = v_2^T A v_1 = v_2^T A^T v_1 = \lambda_2 v_2^T v_1.$$

   Hence $(\lambda_1 - \lambda_2) v_2^T v_1 = 0$. Since $\lambda_1 \neq \lambda_2$, the claim follows.

3. (a) Disproof. If $y(t)$ is a solution, then

$$\frac{d}{dx}(y_1^2 + y_2^2) = 2(y_1 y_1' + y_2 y_2') = 2(y_1 y_2 - y_2 y_1) = 0.$$

   Hence $y_1(t)^2 + y_2(t)^2$ does not depend on x, so is equal to $y_1(0)^2 + y_2(0)^2 = v_1^2 + v_2^2 = \|v\|^2 \neq 0$. I.e., for every x, $y(t)$ lies on a circle of radius $\|v\| \neq 0$, hence (since $y(t)$ is not constant) does not lie on a straight line.

   (b) Proof. $\det(\lambda I - A)$ is a 5th degree polynomial in λ. Odd degree polynomials with real coefficients have at least one real root, i.e., A has at least one real eigenvalue $\hat{\lambda}$. Let $\hat{v}$ be a (real) eigenvector associated to $\hat{\lambda}$. If $y(0) = \hat{v}$ then $y(t) = \exp(tA)\hat{v} = \exp(\hat{\lambda}t)\hat{v}$, which lies on the ray (straight half-line) that supports $\hat{v}$.