# PROBABILITY - Ph.D. Qualifying Exam Fall 2018

**Part 1:** Consider that two random variables $X$ and $Y$ are jointly uniformly distributed in the set $\mathbb{A}$ depicted in Figure 1, or equivalently, the joint probability density function (PDF) is:

$$f_{XY}(x,y) = \begin{cases} 0.5 & \text{if } y \le x \text{ and } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

A) **(4 pt)** Obtain $f_Z$ denoting the PDF of $Z := X^2$.

B) **(3 pt)** Determine $f_W$ representing the PDF of $W := Y - X$.

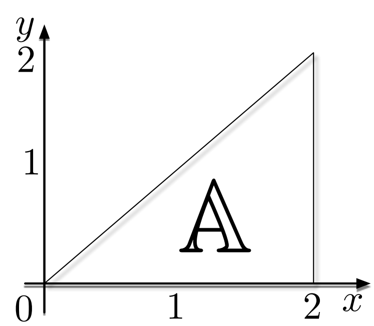


Figure 1: The joint PDF $f_{X,Y}$ is uniformly distributed in $\mathbb{A}$ and is $0$ outside of $\mathbb{A}$.

**Exercise 2:** Assume that a waiter receives credit card payments from $n$ customers and accidentally mixes up the cards in a totally random way before returning them.

A) **(3 pts.)** Determine the probability that at least one customer will get the right card when $n$ is $4$.

B) Let $P_n$ represent the probability that no customer gets the right card when there are $n$ customers.

B-i) **(2 pts.)** Determine a sequence of non-negative constants $\alpha_1, \alpha_2, \dots$ such that $P_n$ satisfies the following equality for every $n$ greater than or equal to $3$:

$$P_n + \alpha_1 P_{n-1} + \cdots + \alpha_{n-2} P_2 + \alpha_{n-1} P_1 + \alpha_n = 1,$$

B-ii) **(2 pts.)** Could $P_n$ converge to $0$ or $1$ as $n$ tends to infinity? Briefly justify your answer.

**Exercise 3:** Consider a system that comprises $N$ components, and let $X_1$ through $X_N$ represent their lifetimes. Assume that $X_1$ through $X_N$ are all independent exponential random variables with parameter $\lambda$.

A) **(3 pts.)** Let $Z_{min} := \min\{X_1, \dots, X_N\}$ be the random variable defined as the time at which the first component fails. Determine the PDF and the expected value of $Z_{min}$ as a function of $N$ and $\lambda$.

B) **(3 pts.)** Let $Z_{max} := \max\{X_1, \dots, X_N\}$ be the random variable defined as the time at which the last component fails. Determine the PDF of $Z_{max}$ as a function of $N$ and $\lambda$.

**SOLUTIONS**

**SOLUTION 1-A)**

$$f_X(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(Z \le z) = P(X^2 \le z) = P(X \le \sqrt{z}), \quad 0 \le z \le 4$$

$$P(X \le \sqrt{z}) = \begin{cases} \frac{z}{4} & 0 < z \le 4 \\ 0 & z \le 0 \\ 1 & z > 4 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{4} & 0 < z \le 4 \\ 0 & \text{otherwise} \end{cases}$$

**SOLUTION 1-B)**

$$P(W \le w) = \begin{cases} 1 & w > 0 \\ \frac{(2+w)^2}{4} & -2 \le w \le 0 \\ 0 & w < -2 \end{cases}$$

$$f_W(w) = \begin{cases} 1 + \frac{w}{2} & -2 \le w \le 0 \\ 0 & \text{otherwise} \end{cases}$$

**SOLUTION 2-A)** Using counting arguments $P(\text{no right card}) = \frac{9}{24}$ so $P(\text{at least one card match}) = \frac{15}{24}$.

**SOLUTION 2-B-i)**

$$\alpha_n = \frac{1}{n!}, \quad n \ge 1$$

To get to this result notice that the following identity concerning the number of permutations holds:

$n! = \#(n \text{ cards match, n clients}) + \#(n-1 \text{ cards match, n clients}) + \cdots + \#(0 \text{ cards match, n clients})$

Define $G_q = \#(0 \text{ cards match, q clients})$ for $q \ge 1$, then for $1 \le m < n$ we have

$$\#(m \text{ cards match, n clients}) = \binom{n}{m} G_{n-m}$$

Dividing the first identity by $n!$ and from the above, we get:

$$1 = \frac{1}{n!} + \binom{n}{n-1}\frac{G_1}{n!} + \binom{n}{n-2}\frac{G_2}{n!} + \cdots + \binom{n}{1}\frac{G_{n-1}}{n!} + \frac{G_n}{n!}$$

$$1 = \frac{1}{n!} + \frac{1}{(n-1)!}G_1 + \frac{1}{(n-2)!}\frac{G_2}{2!} + \cdots + \frac{G_{n-1}}{(n-1)!} + \frac{G_n}{n!}$$

We conclude by noticing that $P_n = \frac{G_n}{n!}, \quad n \ge 1$.

**SOLUTION 2-B-ii)** Clearly the limit cannot be $0$ nor $1$. The limit is $e^{-1}$.

**SOLUTION 3-A)**

$$P(Z_{min} > z) = P(X_1 > z) \cdots P(X_N > z) = \begin{cases} e^{-N\lambda z} & z \ge 0 \\ 0 & z < 0 \end{cases}$$

This gives $f_{Z_{min}}(z) = N\lambda e^{-N\lambda z}$ for $z \ge 0$ and $E[Z_{min}] = \frac{1}{N\lambda}$.

**SOLUTION 3-B)** Using a similar method $P(Z_{max} \le z) = [1 - e^{-\lambda z}]^N$ for $z \ge 0$ and $f_{Z_{max}}(z) = \lambda N[1 - e^{-\lambda z}]^{N-1}e^{-\lambda z}$ for $z \ge 0$.