Basic Mathematics - Ph.D. Written Qualifying Examination, Spring 2018

(1) (7 pts.) Find the general solution of the following simultaneous differential equations:

(a) $\frac{dx}{dt} = x - 2y, \ \frac{dy}{dt} = y - 2x$ **(3 pts.)**

(b) $\frac{d^2x}{dt^2} = x - y, \ \frac{d^2y}{dt} = y - x$ **(4 pts.)**

(2) (6 pts.) Evaluate the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x^2 dxdy}{\left(1+\sqrt{x^2+y^2}\right)^5}$$

(3) (7 pts.) Let $\hat{\mathbf{u}}$ be a unit vector on the Cartesian plane. For an arbitrary vector $\mathbf{v}$, let $\mathbf{w}$ be the reflection (mirror image) of $\mathbf{v}$ across the line containing $\hat{\mathbf{u}}$. Show that

$$\mathbf{w} = 2(\hat{\mathbf{u}} \bullet \mathbf{v})\hat{\mathbf{u}} - \mathbf{v}$$

Writing $\mathbf{w} = \mathbf{R}\mathbf{v}$ and taking the Cartesian coordinates of $\hat{\mathbf{u}}$ and $\mathbf{v}$ to be respectively (u_1,u_2) and (v_1,v_2), find the components of the matrix $\mathbf{R}$. Verify that $\mathbf{R}^2 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Solutions

(1) (a) From first equation $y = \frac{x}{2} - \frac{1}{2}\frac{dx}{dt}$. Substitute in second equation $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0$.

This can be written as $D^2 - 2D - 3 = (D-3)(D+1)$. $D = 3,-1$

General solution is $x = Ae^{-t} + Be^{3t}$ and $y = Ae^{-t} - Be^{3t}$

(b) From first equation $y = x - \frac{d^2x}{dt^2}$. Substitute in second equation $\frac{d^4x}{dt^4} - 2\frac{d^2x}{dt^2} = 0$

$D^4 - 2D^2 = D^2(D^2 - 2) = 0$. $D = \pm\sqrt{2}, 0$. General solution is $x = At + B + Ce^{\sqrt{2}t} + De^{-\sqrt{2}t}$

$y = At + B - Ce^{\sqrt{2}t} - De^{-\sqrt{2}t}$

(2) Substitute $x = r\cos(\theta), y = r\sin(\theta)$ switch infinite domain of integration to polar coordinates.

Integral becomes $\int_0^\infty \int_0^{2\pi} \frac{r^3\cos^2(\theta)\,dr\,d\theta}{(1+r)^5}$ = $\int_0^\infty \int_0^{2\pi} \frac{r^3(1+\cos(2\theta))\,dr\,d\theta}{2(1+r)^5}$ = $\pi\int_0^\infty \frac{r^3}{(1+r)^5}dr$

Substitute $1 + r = x$. Integral becomes $\pi\int_1^\infty \frac{(x-1)^3}{x^3}dx = \pi\int_1^\infty \frac{1}{x^2} - \frac{3}{x^3} + \frac{3}{x^4} - \frac{1}{x^5}dx$

= $\left[\pi\left(-\frac{1}{x} + \frac{3}{2x^2} - \frac{1}{x^3} + \frac{1}{4x^4}\right)\right]_1^\infty = \frac{\pi}{4}$

(3) $\mathbf{v} \bullet \hat{\mathbf{u}} = |\mathbf{v}|\cos(\theta)$ where θ is the angle between the two vectors. Also $\mathbf{w} \bullet \hat{\mathbf{u}} = |\mathbf{w}|\cos(\theta)$. All three vectors are in the same plane.

$\mathbf{v} + \mathbf{w} = 2|\mathbf{v}|\hat{\mathbf{u}}\cos(\theta)$ gives $\mathbf{w} = 2(\hat{\mathbf{u}} \bullet \mathbf{v})\hat{\mathbf{u}} - \mathbf{v}$ Q.E.D.

Substitute $\hat{\mathbf{u}}, \mathbf{v}$ as column vectors $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ note that $u_1^2 + u_2^2 = 1$ gives

$\mathbf{R} = \begin{pmatrix} u_1^2 - u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_2^2 - u_1^2 \end{pmatrix}$ and it follows that $\mathbf{R}^2 = \mathbf{I}$. **Q.E.D.**