Electromagnetics – Ph.D. Written Qualifying Examination, Spring 2018

Problem 1 (6 pts)
There are two concentric conducting cylinders that are infinitely long. The inner radius is *a* and the outer radius is *b*. The electric potential is V(*a*) = 0 V and V(*b*) = V_o. The space between the conductors is filled with air AND a uniform volume charge density ρ_o (C/m^3). Find V(r) between the conducting cylinders.

Problem 2 (6 pts)
A disk with a radius of *b* = 5 cm is centered at the origin in the z=0 plane. The disk has a surface current density of $\vec{J}_s = 2\cos(\varphi)\ \hat{\varphi}\,(\text{A/m})$. Find the magnetic field intensity 10 cm above the center of the disk (at z = 10 cm; x=0=y).

Problem 3 (3 pts)
What is the Lorentz gauge and why is it useful / important?

Problem 4 (5 pts)
A generic waveguide (a =2 cm; b = 1 cm) is filled with a lossy dielectric with $\varepsilon_r = 2 + 0.1j$. What is the cutoff frequency of the fundamental mode in the waveguide and what is the attenuation of the waveguide (in dB/m) due **only** to the dielectric when a wave is traveling in the guide at a frequency 50% higher than the fundamental cutoff frequency?

You may find these formulas useful:

$$\text{Cylindrical coordinates:}\ \vec{\nabla}V = \hat{a}_r \frac{\partial V}{\partial r} + \hat{a}_\varphi \frac{1}{r}\frac{\partial V}{\partial \varphi} + \hat{a}_z \frac{\partial V}{\partial z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}; \quad \vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{a}_r & r\hat{a}_\varphi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$$

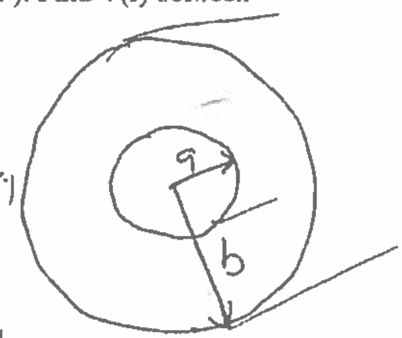
$$\nabla^2 V = \frac{1}{r}\frac{\partial (r\partial V/\partial r)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

Electromagnetism Solutions.

or $V(r) = V_0 + \frac{(b^2-r^2)\rho_0}{4\varepsilon_0} - \left[V_0 + \frac{(b^2-a^2)\rho_0}{4\varepsilon_0}\right]\frac{\ln(r/b)}{\ln(a/b)}$

Problem 1 (~~10~~ 6 pts)

There are two concentric conducting cylinders that are infinitely long. The inner radius is a and the outer radius is b. The electric potential $V(a) = 0$ V and $V(b) = V_o$. The space between the conductors is filled with air AND a uniform volume charge density ρ_o (C/m^3). Find V(r) between the conducting cylinders.



$\nabla^2 V = -\rho_0/\varepsilon_0 = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)$ since V cannot depend on φ or z due to symmetry

$\Rightarrow r\frac{dV}{dr} = -\frac{r^2\rho_0}{2\varepsilon_0} + C_1$ & $\vec{E} = -\frac{dV}{dr} = \frac{r\rho_0}{2\varepsilon_0} - \frac{C_1}{r}$

$\Rightarrow \boxed{V(r) = -\frac{r^2\rho_0}{4\varepsilon_0} + C_1\ln(r/a) + C_2}$ 3 points to here

$V(a) = 0 \Rightarrow -\frac{a^2\rho_0}{4\varepsilon_0} + C_2 = 0$ or $C_2 = \frac{a^2\rho_0}{4\varepsilon_0}$ +2

& $V(r) = \frac{(a^2-r^2)\rho_0}{4\varepsilon_0} + C_1\ln(r/a)$

$V(b) = V_0 = \frac{(a^2-b^2)\rho_0}{4\varepsilon_0} + C_1\ln(b/a)$

$C_1 = \left[V_0 - \frac{(a^2-b^2)\rho_0}{4\varepsilon_0}\right]/\ln(b/a)$ +2

or $\boxed{V(r) = \frac{(a^2-r^2)\rho_0}{4\varepsilon_0} + \left[V_0 - \frac{(a^2-b^2)\rho_0}{4\varepsilon_0}\right]\frac{\ln(r/a)}{\ln(b/a)}}$

X.C. $\vec{E}(a) = \frac{a\rho_0}{2\varepsilon_0} - \frac{C_1}{a} = 0 \Rightarrow C_1 = \frac{a^2\rho_0}{2\varepsilon_0} = \frac{2a^2\rho_0}{4\varepsilon_0}$

Extra Credit (3 points) Find an expression for ρ_o so that the electric field is zero at $r = a$.

or $V_0 - \frac{(a^2-b^2)\rho_0}{4\varepsilon_0} = \frac{2a^2\rho_0}{4\varepsilon_0}\ln(b/a)$

or $V_0 = \frac{\rho_0}{4\varepsilon_0}\left[2a^2\ln(b/a) + a^2 - b^2\right]$

or $\boxed{\rho_0 = 4V_0\varepsilon_0/\left[2a^2\ln(b/a) + a^2 - b^2\right]}$

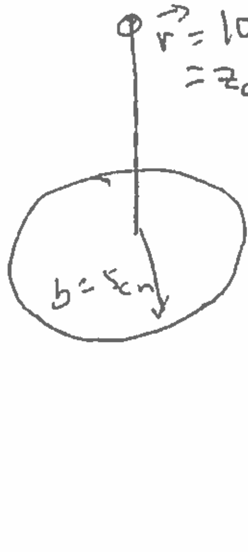
6

Problem 2([redacted] pts)

A disk with a $b = 5$ cm radius is centered at the origin in the z=0 plane. The disk has a surface current density of $\vec{J}_s = $ [redacted] $\cos(\varphi)\,\hat{\varphi}$ (A/m). Find the magnetic field intensity 10 cm above the center of the disk (at z = 10 cm; x=0=y).

$z_0\hat{z}$

$\vec{r} = 10\hat{z}$ cm $= z_0\hat{z}$



$b = 5$ cm

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int_S \frac{\vec{J}_s \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,ds$$

$\vec{r} = 10\hat{z}$ cm

$\vec{r}\,' = r'\hat{r}$

$\vec{J}_s = $ [redacted] $\cos\phi'\,\hat{\varphi}$ A/m

$ds = r'dr'd\phi'$

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^b r'dr' \frac{[\text{redacted}]\cos\phi'\,\hat{\varphi}\times(z_0\hat{z} - r'\hat{r})}{(z_0^2 + r'^2)^{3/2}}$$

$\hat{\varphi}\times\hat{z} = \hat{r}$

$= \hat{x}\cos\phi' + \hat{y}\sin\phi'$

$\hat{\varphi}\times(-\hat{r}) = \hat{z}$

$$\vec{H}(\vec{r}) = \frac{z_0 [\text{redacted}]}{4\pi}\int_0^{2\pi} d\phi' \cos\phi'(\hat{x}\cos\phi' + \hat{y}\sin\phi')\int_0^b \frac{r'dr'}{(z_0^2+r'^2)^{3/2}}$$

$$+\ \underbrace{\frac{\hat{z}[\text{redacted}]}{4\pi}\int_0^{2\pi} d\phi'\cos\phi'}_{=0}\int_0^b \frac{r'^2dr'}{(z_0^2+r'^2)^{3/2}}$$

$$\vec{H}(\vec{r}) = \frac{[\text{redacted}]}{4\pi}z_0 \underbrace{\int_0^{2\pi} d\phi'(\cos^2\phi'\,\hat{x} + \cancel{\sin\phi'\cos\phi'\,\hat{y}})}_{=\pi\hat{x}} \int_0^b \frac{r'dr'}{(z_0^2+r'^2)^{3/2}}$$

$=0$

$u = r'^2 + z_0^2$

$du = 2r'dr'$

$r' = 0 \Rightarrow u = z_0^2$

$r' = b \Rightarrow u = b^2 + z_0^2$

$$\vec{H}(\vec{r}) = \frac{[\text{redacted}]}{2}\hat{x}\int_0^b \frac{z_0 r'dr'}{(z_0^2+r'^2)^{3/2}}$$

$$\vec{H}(\vec{r}) = \frac{[\text{redacted}]}{2}\hat{x}\int_{z_0^2}^{b^2+z_0^2}\left(\frac{z_0}{2}\right)u^{-3/2}du$$

$$\vec{H}(\vec{r}) = \frac{[\text{redacted}]}{2}\hat{x}\,\frac{z_0}{2}\,\frac{u^{-1/2}}{-1/2}\Bigg|_{z_0^2}^{b^2+z_0^2} = -\frac{[\text{redacted}]}{2}\hat{x}\,z_0\left(\frac{1}{\sqrt{b^2+z_0^2}} - \frac{1}{\sqrt{z_0^2}}\right)$$

$$\vec{H}(\vec{r}) = \frac{[\text{redacted}]}{2}\hat{x}\left(1 - \frac{1}{\sqrt{1+(b/z_0)^2}}\right) = \frac{[\text{redacted}]}{2}\hat{x}\left(1 - \frac{1}{\sqrt{1.25}}\right)$$

$\left[\vec{H}(\vec{r}) = 0.053\text{ A/m}\ \ (0.053\text{ A/m})\text{ if you had a calculator}\right]$

Problem #3

Lorentz Gauge : $\vec{\nabla} \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t} = 0$

Decouples differential equations for vector ($\vec{A}$) and scaler (V) potentials.

Problem #4 $\quad f_{c_{10}} = \frac{c}{2a} = \frac{30 \times 10^9 \ \frac{cm}{s}}{\sqrt{2}(2 \times 2)cm} = \frac{7.5}{\sqrt{2}} GHz$

$$\alpha_d = \frac{8.686 \ \omega \tan\delta}{2c\sqrt{1-\omega_c^2/\omega^2}} = \frac{8.686\left(\frac{\omega}{\omega_c}\right)\left(\frac{0.1}{2}\right)\omega_c}{2 \times 3 \times 10^8 \sqrt{1-(\omega_c/\omega)^2}}$$

$$\alpha_d = \frac{8.686(1.5)(.05)\frac{\cancel{c}}{2a}2\pi}{2 \times \cancel{c}\sqrt{1-(1/1.5)^2}} = \frac{8.686(1.5)2\pi}{80(.02)\sqrt{1-(1/1.5)^2}} \ \frac{dB}{m}$$

$$\alpha_d = 8.686\left(\frac{1.5}{1.6}\right)\frac{6\pi}{\sqrt{5}} \ \frac{dB}{m}$$

(or $68.04 \ dB/m$ if calculators had been allowed)