Linear Systems and Signals — Ph.D. Qualifying Exam, January 2018

Part 1 (6 pts.)

Consider the linear-time invariant system (operating in discrete time) whose impulse response sequence is given by

$$h[n] \;=\; \delta[n] - 5\delta[n-1] + 4\delta[n-2] - 5\delta[n-3] + 3\delta[n-4]$$

If the input $x[n]$ to the system is periodic with period equal to four samples, is the output $y[n]$ also periodic? If so, what is the fundamental (i.e., least) period, in samples?

Part 2 (7 pts.)

Let

$$s(t) \;=\; \sum_{k=-\infty}^{\infty} S_k e^{jkt}$$

be the standard complex Fourier series expansion of a periodic signal (of period equal to 2π) in continuous time.

(2a) (4 pts.) If $S_k \;=\; 3^{-|k|}$ for all indices k, show that

$$s(t) \;=\; \frac{c}{a + b\cos t}$$

and determine the (real) values a, b and c.

(2b) (3 pts.) With a, b and c as found above, determine the complex Fourier series coefficients $\{V_k\}$ of the signal

$$v(t) \;=\; \frac{2c\sin t}{a + b\cos t}\,,$$

simplifying your answer as much as possible.

Part 3 (7 pts.)

Consider the *causal* linear time-invariant system described by the differential equation

$$3\frac{d^2y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 3y(t) \;=\; \frac{dx(t)}{dt} - 2\,x(t)$$

(3a) (3 pts.) Determine the zeros and poles of the system's transfer function $H(s)$. Is the system stable in the bounded-input, bounded-output (BIBO) sense?

(3b) (4 pts.) Let the input to the system be the negative-time signal

$$x(t) \;=\; e^{2t}u(-t)$$

Determine the resulting output $y(t)$. Does the product $x(t)y(t)$ have a particularly simple form?

Part 1

If the input to a LTI system is periodic, then so is the output. The fundamental period of y[n] is either the same as, or a submultiple of, that of x[n].

In this case, the period of x[n] is four samples. Letting

x[0:3] = (a,b,c,d),

we have

y[0] = a - 5*d + 4*c - 5*b + 3*a = 4*a - 5*d + 4*c - 5*b

y[1] = b - 5*a + 4*d - 5*c + 3*b = -5*a + 4*b - 5*c + 4*d

y[2] = c - 5*b + 4*a - 5*d + 3*c = 4a - 5*d + 4*c - 5*b

y[3] = d - 5*c + 4*b - 5*a + 3*d = -5*a + 4*b - 5*c + 4*d

Since y[0] = y[2] and y[1] = y[3], the output has fundamental period equal to two samples.

Part 2

(2a)

s(t) = -1 + sum{3^(-k)*exp(j*k*t) over k>=0}

+ sum{3^(-k)*exp(-j*k*t) over k>=0}

= -1 + inv(1 - exp(j*t)/3) + inv(1 - exp(-j*t)/3)

= -1 + 2*(1 - cos(t)/3)/(1 + 1/9 - 2*cos(t)/3)

= 4/(5-3*cos(t))

(2b)

v(t) = 2*s(t)*sin(t)

= (-j)*s(t)*exp(j*t) + j*s(t)*exp(-j*t)

Therefore

V[k] = (-j)*(S[k-1] - S[k+1])

= j*(3^(-|k+1|) - 3^(-|k-1|))

i.e.,

V[k<0] = -j*(8/3)*(3^k)

V[0] = 0

V[k>0] = j*(8/3)*(1/3)^k

Part 3
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(3a)
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H(s) = (s-2)/(3*s^2 + 10*s + 3)

Zeros at s = 2, |s| = infty; poles at s = -1/3, -3

System is causal, hence ROC(H) : Real{s} > -1/3

ROC includes imaginary axis, so system is BIBO stable

(3b)
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X(s) = -1/(s-2) ,   ROC(X) : Real{s} < 2

ROC(H) and ROC(X) have nonempty intersection, hence x(t) is acceptable as input.

Y(s) = X(s)*X(s) = -1/(3*s^2 + 10*s + 3)

with ROC(H) : Real{s} > -1/3 (note pole-zero cancelation).

Y(s) = (1/8)/(s+3) - (1/8)/(s+1/3)

thus

y(t) = (1/8)*(exp(-3*t)-exp(-t/3))*u(t)

Note that x(t)*y(t) = 0 except (trivially) at t = 0.