PROBABILITY - Ph.D. Qualifying Exam Spring 2018

Part 1: Consider that a message with N bits is stored in a noisy memory device that flips each bit with probability p. Flip events are independent across bits. Let X be the random variable that quantifies the total number of bit flips.

A) (3 points) Assume that $E[X] = 5$ and $N = 10$, where $E[X]$ represents the expectation or mean value of X. Determine $P(X = 2)$ and $E[X^2]$.

B) (4 points) Suppose that $N = 10^3$ and that information can be correctly recovered from the memory when at most 2 bits are flipped. An error occurs when 3 or more bits are flipped. The cost of a memory chip is given by $2 \times 10^3 e^{-Np}$ and the cost of making an error is 10^3. Choose p that minimizes the expected total cost given by $10^3(P(error) + 2e^{-Np})$. (Hint:You should resort to the Poisson-approximation to solve this problem.)

Part 2: Consider that a bucket contains N components that are labeled from 1 to N. Components are taken uniformly randomly from the bucket without replacement. Consider that you have taken t components and placed them in a shopping bin.

A) (4 points) What is the probability that the component with label 1 is in the bin ? Give a formula that depends on t and N, for $1 \leq t \leq N$.

B) (2 points) What is the probability that all the components in the bin have label less than or equal to t ? Give a formula that depends on t and N, for $1 \leq t \leq N$.

Part 3: Let W_1 and W_2 be two independent random variables that are uniformly distributed between -1 and 1.

A) (4 points) Sketch and determine explicitly $P(W_1 + \alpha \geq W_2)$ as a function of α.

B) (3 points) Assume that $Z = \max\{W_1, W_2\}$. Determine the probability density function of Z and compute $E[Z]$.

SOLUTIONS

SOLUTION 1-A) $p = 0.5$, $P(X = 2) = 45 \times \frac{1}{1024}$, $Var[X] = 10 \times 0.5^2$, $E[X^2] = Var[X] + E[X]^2 = 27.5$

SOLUTION 1-B) The cost can be written as $10^3(1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) + 2e^{-\lambda})$ in terms of $\lambda = np$. Take derivative to get $10^3 e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2} - (\lambda + 1) - 2)$. The derivative is zero when $\lambda = 2$, or equivalently $p = 2 \times 10^{-3}$.

SOLUTION 2-A) $\dfrac{\binom{N-1}{t-1}}{\binom{N}{t}} = \frac{t}{N}$

2-B) $\dfrac{1}{\binom{N}{t}}$

SOLUTION 3-A) $P(W_1 + \alpha \geq W_2) = \begin{cases} 0 & \alpha \leq -2 \\ \frac{(\alpha+2)^2}{8} & -2 < \alpha \leq 0 \\ 1 - \frac{(\alpha-2)^2}{8} & 0 < \alpha \leq 2 \\ 1 & \alpha > 2 \end{cases}$

SOLUTION 3-B)

$$P(Z \leq z) = P(W_1 \leq z)P(W_2 \leq z) = \begin{cases} 0 & z < -1 \\ 1 & z \geq 1 \\ \frac{1}{4}(z+1)^2 & \text{otherwise} \end{cases}$$

So, the PDF is the derivative of the CDF:

$$f_Z(z) = \begin{cases} 0 & |z| \geq 1 \\ \frac{1}{2}(z+1) & \text{otherwise} \end{cases}$$

So, $E[z] = 0.5 \int_{-1}^{1} z^2 + z dz = 0.5(\frac{1}{3}z^3 + \frac{1}{2}z^2)|_{-1}^{1} = \frac{1}{6}(1 - (-1)) = \frac{1}{3}$