Spring 2019 Written Qualifying Examination

Basic Mathematics

1. (7 pts) The following questions are to be tackled *without making use of properties of the exponential function.* (You are allowed to use the Taylor-series definition of the exponential function, as well as those of the sin and cos function, if you need them.)

   (a) (4 pts) Let k be an integer and $\varphi(k)$ be a function of k such that $\varphi(k) \to +\infty$ as $k \to +\infty$. Does the scalar sequence $\{x_k\}_{k=0}^{\infty}$, where

   $$x_k = \big(1 + 2\varphi(k) + 3(\varphi(k))^2 + 4(\varphi(k))^3 + 5(\varphi(k))^4\big)\, \mathrm{e}^{-\varphi(k)},$$

   necessarily converge (to a finite limit) as $k \to +\infty$, and if it does, what is the limit? Justify your steps.

   (b) (3 pts) Prove the identity

   $$\mathrm{e}^{i\pi} + 1 = 0,$$

   where $i^2 = -1$.

2. (6 pts)
   Let M be an $m \times n$ real matrix such that, for all $x \in R^n$, $\|Mx\| = \|x\|$, where $\|\cdot\|$ denotes the Euclidean norm of its vector argument; e.g., $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots, x_n^2}$. The following proofs must be carried out using **only** the rules of vector and matrix multiplication. (If you wish to use other properties, you must first prove them.)

   (a) (2 pts) Prove that M has full column rank (i.e., that its columns are linearly independent).

   (b) (4 pts) Provide a detailed proof that $M^T M = I$.

3. (7 pts)

   (a) (2 pts) Give four linearly independent solutions to the equation

   $$\frac{\mathrm{d}^4 y(t)}{\mathrm{d}t^4} = y(t).$$

   (b) (5 pts) Consider the second-order differential equation

   $$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = \frac{n}{t}\frac{\mathrm{d}y(t)}{\mathrm{d}t},$$

   where n is a positive integer. (Clearly, constants (e.g., $y(t) = 1\ \forall t$) are solutions.) Find a non-constant solution. Justify your steps.

Solutions

1. (a)
$$e^t > 1 + t + (1/2!)t^2 + (1/3!)t^3 + (1/4!)t^4 + (1/5!)t^5 > (1/5!)t^5 \quad \forall t > 0,$$
so that, (for k large enough that $\varphi(k) > 0$),
$$0 < x_k < \left(1 + 2\varphi(k) + 3(\varphi(k))^2 + 4(\varphi(k))^3 + 5(\varphi(k))^4\right)(\alpha(\varphi(k))^{-5}),$$
where $\alpha = 5!$. Since $\varphi(k) \to \infty$ as $k \to \infty$, x_k goes to zero as $k \to \infty$.

   (b)
$$e^i = 1 + it - (1/2!)t^2 - (1/3!)it^3 + (1/4!)t^4 + \cdots = \cos(t) + i\sin(t),$$
so that $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

2. (a) Suppose $x \neq 0$. Then $\|x\| \neq 0$ and, since $\|Mx\| = \|x\|$, $Mx \neq 0$ i.e., M has full column rank.

   (b) $x'M'Mx = \|Mx\|^2 = \|x\|^2 = x'x$ Hence, with $S = M'M - I$, $x'M'Mx - x'x = x'Sx = 0$ for all x. In particular, given a standard basis vector $e_i \in R^n$, $S_{ii} = e_i{}'Se_i = 0$ for all i, and $S_{ij} + S_{ji} = S_{ii} + S_{jj} + S_{ij} + S_{ji} = (e_i + e_j)'S(e_i + e_j) = e_i{}'Se_j + e_j{}'Se_i = 0$ for all i, j. Since S is symmetric, we conclude that $S = 0$, i.e., $M'M = I$.

3. (a) $\sin(t)$, $\cos(t)$, e^t and e^{-t}.

   (b) Let $z = y'$. Then
$$z'(t) = \frac{n}{t}z(t),$$
i.e.,
$$\frac{dz}{z} = n\frac{dt}{t},$$
yielding (up to an additive constant)
$$\log z(t) = n \log t$$
i.e., $z(t) = t^n$. Hence $y(t) = \frac{t^{n+1}}{n+1}$ is a solution.