Problem 1

(a) \( \Omega \) is formed of \( 2^n \) outcomes, one for each of the possible results of \( n \) tosses. 
\( F \) consists of all subsets of \( \Omega \); thus, every subset \( A \) is an event.

Given a subset \( A \subset \Omega \), 
\[
P(A) = \sum_{\omega \in A} P(\omega) = \sum_{\omega \in A} 2^{-n}.
\]

\[
P(A) = |A| \cdot \frac{1}{2^n}
\]

(b) \( \Omega \) is formed of all infinite sequences of coin tosses, i.e., of all

(i) infinite sequences of binary digits. The cardinality of \( \Omega \) is the same

as that of \([0,1] \) or \([0,1]\), and these are the same as \( \mathbb{R} \).

(ii) \( P(\omega) = 0 \) because if \( P(\omega) > 0 \), this would violate the requirement \( P(\Omega) = 1 \).

(b) Following the presentation in class

\[
P\left( \omega : d(\omega_0) = d(\omega_1) = \ldots = d(\omega_{10}) = 1 \right) = P\left( \sum_{i=1}^{10} \frac{1}{2^i} , \sum_{i=1}^{10} \frac{1}{2^i} + \frac{1}{2^{10}} \right) = \text{length of the interval} = 2^{-10}
\]

(c) Let \( A \) be the event described in the question.

Conditional on \( A \), the proportion of 1's and 0's in the odd digits will be \( \frac{1}{2} \) by SLLN.

Thus the elements \( \omega \in A \) will have \( \frac{3}{4} \) ones and \( \frac{1}{4} \) zeros \( \omega \) with prob 1.

However, these \( \omega \)'s lie in the complement \( N^c \) of the set of normal numbers, and \( P(N^c) = 0 \). Answer: \( P(A) = 0 \)

Thus \( A \subset N^c \)

\( P(A) \leq P(N^c) = 0 \).
Problem 2

(a) \( n = 3 \). The only possible outcome that satisfies the condition is when one of the boxes contains 3 balls and 2 boxes are empty. The probability of this equals
\[
P_3 = \binom{n}{2} \left( \frac{1}{n} \right)^n = \binom{3}{2} \left( \frac{1}{3} \right)^3 = \frac{1}{9}
\]

(b) \( n = 4 \). Now there are two possibilities:
- \( A_n^{(1)} \): one of the boxes contains one ball, and one contains 3 balls
- \( A_n^{(2)} \): two of the 4 boxes contain 2 balls each.

Let us compute \( P(A_n^{(1)}) \):
there are \( \binom{4}{2} \) choices of 2 empty boxes; there are 2 choices of the box with 3 balls out of the boxes that contain balls; there are 4 each choices of the ball that has a box to itself.
\[
P(A_n^{(1)}) = \binom{4}{2} \times 2 \times 4 \times \left( \frac{1}{4} \right)^4 = \frac{12}{64}
\]
For \( A_n^{(2)} \): There are \( \binom{4}{2} \) choices of 2 empty boxes; \( \frac{1}{2} \binom{4}{2} \) ways to place 2 pairs in the boxes with 2 balls; then so we obtain:
\[
P(A_n^{(2)}) = \binom{4}{2} \times \frac{1}{2} \binom{4}{2} \times \frac{2}{4} = \frac{9}{64}
\]
Answer: \( p_4 = P(A_n^{(1)}) + P(A_n^{(2)}) = \frac{21}{64} \).

(c) Let \( n \geq 5 \). Again there are only two options:
- \( A_n^{(1)} \): One of the boxes contains 3 balls, \( n - 3 \) contain one ball each
- \( A_n^{(2)} \): 2 boxes contain 2 balls each, \( n - 4 \) contain a single ball each.

For \( A_n^{(1)} \): \( \binom{n}{2} \) choices of the empty boxes; \( \frac{n}{2} \binom{n-2}{2} \) ways to
\( n \) ways to select the box that contains 3 balls;
\( \binom{n}{3} \) ways to choose which 3 balls occupy the box that they share.
Probability that the balls are placed as desired is
\[
\binom{n-3}{n-2} \ldots \binom{1}{n} \times \left( \frac{1}{n} \right)^3
\]
Thus,

\[ P(A_{n}^{(1)}) = \left( \frac{n}{2} \right) (n-2) \left( \frac{n}{3} \right) \frac{(n-3)!}{n^n} \]

For \( A_{n}^{(2)} \): \( \binom{n}{2} \) choices of the empty boxes

\[ \frac{1}{2} \binom{n}{2} \binom{n-2}{2} \text{ ways two place two pairs of balls in the boxes that contain two balls} \]

The probability that \( n-2 \) balls will fall one per box is

\[ \frac{(n-2)!}{n^{n-2}} \]

The probability that the remaining two balls will fall in the designated cells is \( \frac{1}{n^2} \)

Thus

\[ P(A_{n}^{(2)}) = \left( \frac{n}{2} \right) \times \frac{1}{2} \left( \frac{n}{2} \right) \left( \frac{n-2}{2} \right) \frac{(n-2)!}{n^n} = \frac{1}{4} \frac{n!}{n^n} \left( \frac{n}{2} \right) \left( \frac{n-2}{2} \right) \]

Finally,

\[ \Phi_n = P(A_{n}^{(1)}) + P(A_{n}^{(2)}) \neq \text{ (the answer doesn't simplify to a nice expression) } \]
Problem 3

(a) \( \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2} \)

(b) \( H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots \frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k} \)

Let \( n = 2^k \), then

\( H_{2^k} > \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k} > k \cdot \frac{1}{2} \)

(c) Thus \( \lim_{k \to \infty} H_{2^k} = \infty \), i.e., \( \sum_{i=1}^{\infty} \frac{1}{i} = \infty \)

(d) For \( s < 1 \), \( \frac{1}{n^s} > \frac{1}{n} \) for all \( n \in \mathbb{N} \)

Thus \( \sum_{n=1}^{m} \frac{1}{n^s} > \sum_{n=1}^{m} \frac{1}{n} \); since \( \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n} = \infty \), the same is true for \( \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n^s} \). By definition, this implies the claim.

(e) We have \( \frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \ldots + \frac{1}{(kn)^s} < n \cdot \frac{1}{n^s} = \frac{1}{n^t} \)

(f) \( \frac{1}{3^s} + \frac{1}{4^s} < \frac{1}{2^t} \)

\( \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} < \frac{1}{4^s} = \left(\frac{1}{2^t}\right)^2 \)

\( \vdots \)

\( \frac{1}{(2^k+1)^s} + \ldots + \frac{1}{(2^k)^s} < \frac{1}{(2^k)^s} = \left(\frac{1}{2^t}\right)^{k-1} \)

Since \( \frac{1}{2^t} + \frac{1}{(2^t)^2} + \frac{1}{(2^t)^3} + \ldots = \frac{1}{2^t} \cdot \frac{1}{1-\frac{1}{2^t}} \), for any \( m \geq 1 \) we have

\( \sum_{n=1}^{m} \frac{1}{n^3} < 1 + \frac{1}{2^s} + \frac{1}{1-\frac{1}{2^s}} \cdot \frac{1}{2^t} \)

Thus, the sums \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) for increasing \( m \) form an increasing sequence which is bounded above, and therefore has a limit.
(3) \( \sum_{n=1}^{\infty} p_n = \infty \), thus by Borel-Cantelli the required prob. = 1

(2) \( \sum_{n=1}^{\infty} p_n < \infty \), thus by Borel-Cantelli the required prob. = 0

Problem 4

(a) Suppose that player 1 tossed \( k_1 \) Heads and player 2 tossed \( k_2 \) heads.

The probability of this is

\[
\left( \frac{n+1}{k_1} \right)^2 \frac{1}{(n+1)} \left( \frac{n}{k_2} \right)^2 \frac{1}{2^n}
\]

and it depends on \( k_1 \) and \( k_2 \). Thus, the considered outcomes are not equiprobable.

(b) We use the notation \( H_1, T_1, H_2, T_2 \) for the counts of \( H \) and \( T \) of the players.

Clearly \( P(H_1 > H_2) = P(T_1 > T_2) \)

Next, \( H_1 > H_2 \) if and only if \( n - H_1 < n - H_2 \), or \( T_1 < T_2 \), or \( T_1 \leq T_2 \).

Thus \( \{ \omega : H_1 > H_2 \} \subset \{ \omega : T_1 \leq T_2 \} \)

\[ P(T_1 \leq T_2) = P(H_1 > H_2) \]

Further \( P(T_1 > T_2) = 1 - P(T_1 \leq T_2) \)

We conclude:

\[ P(H_1 > H_2) = P(T_1 > T_2) = 1 - P(T_1 \leq T_2) = 1 - P(H_1 > H_2) \]

This equality implies that \( P(H_1 > H_2) = \frac{1}{2} \)