Problem 1. Let $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$. Describe all RVs that are measurable on $(\Omega, \mathcal{F})$.

Problem 2. We say that a $\sigma$-algebra $\mathcal{F}$ is generated by a collection of subsets $\mathcal{A}$ of $\Omega$ if $\mathcal{F}$ is formed of all countable unions and complements of the subsets $F \in \mathcal{A}$.

Let $\Omega = \{1, 2, \ldots, 6\}$, and let $\mathcal{A} = \{\{1, 3, 5\}, \{1, 2, 3\}\}$.

(a) Describe the $\sigma$-algebra $\mathcal{F} = \sigma(\mathcal{A})$ generated by the subsets in $\mathcal{A}$.

(b) Give a list of nonempty elements $G$ of $\mathcal{F}$ such that if $F \in \mathcal{F}$ and $F \subseteq G$, then $F = \emptyset$ or $G$.

Such elements are called atoms of the $\sigma$ algebra. They generate $\mathcal{F}$ constructed in part (a) in the sense that every element in $\mathcal{F}$ can be obtained by taking unions and complements of the atoms and the resulting sets.

(c) Let $\Omega = [0, 1]$ and let $\mathcal{F} = \mathcal{B}([0, 1])$ be the Borel $\sigma$-algebra on $[0, 1]$. What are the atoms of $\mathcal{F}$? Is it true in this case that $\mathcal{F}$ is generated by its atoms (in particular, are open intervals generated by the atoms)?

Problem 3. (i) Let $X$ be a Poisson RV with the parameter $\lambda$, i.e., $p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, 2, \ldots$. Find (a) $E(X)$, (b) $E(X(X-1))$, (c) $\text{Var}(X)$.

(ii) Let $X \sim \mathcal{N}(0, \sigma^2)$. Find (a) $EX^2$, (b) $EX^4$, (c) $E(e^X)$, (d) $E(e^{-X^2})$.

(please no computers, give complete calculations).

Problem 4. Let $N$ be an RV taking values in $\mathbb{Z}_+$, and suppose that $EZ = \mu_1$, $\text{Var}(Z) = \sigma_1^2$. Let $X_1, X_2, \ldots$ be i.i.d. RVs with expectation $\mu_2$ and variance $\sigma_2^2$. Find the mean and variance of the RV $S_N = X_1 + X_2 + \cdots + X_N$. (Hint: You may use the fact that $ES_N = E(E(S_N|N))$.)

Problems 5–8: Solve Problems 1.9, 1.13, 1.17, 1.19 from Bruce Hajek’s book.