Problem 1. (Exercises on convergence)

(a) Give an example of a sequence \( Y_n \) of RVs such that for \( n \to \infty \),

- \( Y_n/n \to 0 \) in probability
- \( Y_n/n^2 \to 0 \) a.s.
- \( Y_n/n \not\to 0 \) a.s.

(b) Let \( X_n, n \geq 1 \) be a sequence of independent RVs with \( P(X_n = 3^n) = P(X_n = -3^n) = \frac{1}{2} \). Let \( S_n = X_1 + \cdots + X_n \).

(i) Compute \( EX_n \) for every \( n \).

(ii) Let \( R_n = \sup \{ t \in \mathbb{R} : P(|S_n| \geq t) = 1 \} \) (the largest number such that \( |S_n| \) is always at least \( R_n \)). Find \( \lim_{n \to \infty} \frac{R_n}{n} \).

(iii) For which \( \epsilon > 0 \) (if any) is it true that \( \lim_{n \to \infty} P(\frac{1}{n}|S_n| \geq \epsilon) \neq 0 \).

(c) Let \( X_n, n \geq 1 \) be a sequence of independent RVs and suppose that \( X_n \xrightarrow{P} X \).

(c1) Prove that for any \( \delta, \delta_1 > 0 \)

\[
P(X - X_n < \delta) > 1 - \delta_1 \quad \text{and} \quad P(X - X_n > -\delta) > 1 - \delta_1.
\]

(c2) Assume toward a contradiction that \( X \) is not constant and choose \( c \) so that \( P(X < c) > 2\epsilon \) and \( P(X > c + \epsilon) > 2\epsilon \). Then show that \( P(X < c) > \epsilon \) and \( P(X > c + \epsilon) > \epsilon \) for a sufficiently large \( n \).

(c3) Now let \( n, m \) be large enough so that \( P(|X_m - X_n| > \epsilon) < \epsilon^3 \). Then argue that

\[
P(|X_m - X_n| > \epsilon) \geq P(X_m < c, X_m > c + \epsilon) = P(X_m < c)P(X_m > c + \epsilon)
\]

and conclude that the assumption leads to a contradiction.

Thus if a sequence of independent RVs converges in probability (and therefore also almost surely), then the limit is a constant with probability one.

(d) Let \( X_n, n \geq 1 \) be independent RVs such that \( X_1 = 0 \) and

\[
P(X_n = n) = P(X_n = -n) = \frac{1}{2n \log n}, \quad P(X_n = 0) = 1 - \frac{1}{n \log n}, \quad n \geq 2.
\]

Let \( S_n = X_1 + \cdots + X_n \). Show that as \( n \to \infty \)

\[
\frac{S_n}{n} \xrightarrow{p} 0 \quad \text{but} \quad \frac{S_n}{n} \not\xrightarrow{a.s.} 0.
\]

(Hint: You will need nothing more than the Borel-Cantelli lemma).

Problem 2. (a) Prove the following extension of Chebyshev’s inequality: let \( \psi \) be a positive-valued function on \( \mathbb{R} \) that is nondecreasing on \( \mathbb{R}_+ \); suppose that \( X \) is an RV such that \( E\psi(X) < +\infty \), then for any \( a \geq 0 \) s.t. \( \psi(a) > 0 \)

\[
P(X > a) \leq \frac{E(\psi(X))}{\psi(a)}
\]

(the version of Chebyshev’s inequality proved in class (also [H], p.20) uses \( \psi(x) = x^2 \).)
(b) Let $X$ be an RV with zero mean and variance $\sigma^2$. Show that

$$P(|X| \geq a) \leq \min\left(\frac{\sigma^2}{a^2}, \frac{2\sigma^2}{a^2 + \sigma^2}\right).$$

(Hint: Use Chebyshev’s inequality in the form shown above in part (a); choose suitable functions $\psi$.)

(c) Suppose that $X \geq 0$ is an RV such that $EX^2 < \infty$ and $EX > t \geq 0$. Prove that

$$P(X > t) \geq \frac{(EX - t)^2}{E(X^2)}.$$

(Hint: Use Cauchy-Schwarz for RVs $X$ and $1_{\{X \geq t\}}$.)

**Problems 3-6.** Solve Problems 2.13, 2.17, 2.19, 2.27 from Bruce Hajek’s book.