Problem 1 (b)

(i) $E X_n = 0$

(ii) $|S_n|$ is the smallest if the realization is $-3, -3^n, ..., -3^{-n}, 3^n$, i.e.,
$$|S_n| \geq \frac{3^n}{2} + \frac{3}{2^n} \text{ w.p. } 1, \quad \lim_{n \to \infty} \frac{R_n}{n} = \infty$$

(iii) For all $\epsilon > 0$, $P\left(\frac{|S_n|}{n} > \epsilon\right) \to 1$

Note that, at the same time, $S_n/n \to 0$ by SLLN

(a) Consider a sequence of RVs $Y_n$ defined on $\Omega = [0, 1]$ and such that
$$Y_n(\omega) = \begin{cases} n & 0 \leq \omega \leq \frac{1}{n} \\ 0 & \text{o/w.} \end{cases}$$

Then $P\left(\frac{|Y_n|}{n} > \epsilon\right) \leq \frac{1}{n} \to 0$, i.e., $\frac{|Y_n|}{n} \to 0$.

Now take $Z_{n,0} = Y_n$, $Z_{n,1} = Y_n + \frac{1}{n}$, $Z_{n,2} = Y_n + \frac{2}{n}$, ..., $Z_{n,n-1} = Y_n + \frac{n-1}{n}$.

The sequence
$$\frac{\sum_{j=0}^{n-1} Z_{n,j}}{n} \to 0$$

at the same time, $\frac{Z_{n,j}}{n} \to 0$ a.s.

However, $\frac{Z_{n,j}}{n^2} \to 0$ a.s.

because as $n$ increases, the sequence $\frac{1}{n^2} Z_{n,j}(\omega) \to 0$ with probability 1.
Prob. 1(c).

Let $\delta \to 0$. Since $X_n \to X$, $P(|X_n - X| > \delta) < \delta$, so for a sufficiently large $n$

$$P(|X_n - X| > \delta) = P((X_n > \delta) \cup (X_n < -\delta)) = P(X_n > \delta) + P(X_n < -\delta) < \delta$$

$\therefore$ $P(X_n < -\delta) > 1 - \delta$ and $P(X_n > \delta) > 1 - \delta$

Now choose $c$ such that $P(X < c) > 2\delta$ and $P(X > c + \delta) > 2\delta$ which is possible unless $X = \text{const}$ a.s.

Then

$$P((X < c) \cap (X_n < X + \delta)) > 2\delta - \delta$$

$$P((X > c + \delta) \cap (X_n > X - \delta)) > 2\delta - \delta$$

From the first of these, for sufficiently large $n$,

$$P(X_n < c + \delta) > P((X < c) \cap (X_n < X + \delta)) > 2\delta - \delta,$$

And since $\delta$ is arbitrarily small,

$$P(X_n < c) > \delta$$

Similarly $P(X_n > c + \delta) > \delta$, both for sufficiently large $n$.

Now let $n,m$ be large enough so that $P(|X_n - X_m| > \delta) < \delta^3$.

Thus

$$\delta^3 > P(|X_n - X_m| > \delta) \geq P(X_n < c, X_m > c + \delta) \geq P(X_n < c)(X_m > c + \delta) \geq \delta^2.$$

The obtained contradiction concludes the proof.
Problem 1 (a) We have $EX_n = 0$; $Var(X_n) = EX_n^2 = 2/n^2 = \frac{1}{2n \ln n} = \frac{n}{2 \ln n}$, $n \geq 2$.

$ES_n = 0$; $Var(S_n) = \sum_{i=2}^{n} \frac{i}{\log i}$

To prove that $\frac{S_n}{n} \to 0$, we can use Chebyshev's inequality. Let us show that $\frac{Var(S_n)}{n^2} \to 0$; this will follow from the next line.

$$\sum_{i=2}^{n} \frac{i}{\log i} = \sum_{i=2}^{\sqrt{n}} \frac{i}{\log i} + \sum_{i=\sqrt{n}+1}^{n} \frac{i}{\log i} \leq \sum_{i=2}^{\sqrt{n}} \frac{1}{\log i} + \sum_{i=\sqrt{n}+1}^{n} \frac{1}{\log \sqrt{n} \cdot i} \leq \frac{\sqrt{n}}{\log \sqrt{n}} + \frac{n}{\log n} \approx \frac{n}{\log n}$$

Then $P(|S_n| > n \varepsilon) \leq \frac{\text{Var}(S_n)}{n^2 \varepsilon^2} \to 0$, proving that $\frac{S_n}{n} \to 0$.

At the same time, let $A_n = \{ \omega : |X_n(\omega)| > n \}$, then

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty$$

so $P(A_n \text{ i.o.}) = 1$, i.e., $|X_n| \to 1$ i.o. At the same time, if $\frac{S_n}{n} \to 0$, then with prob. 1, the event $\frac{|S_n|}{n} < \varepsilon$ occurs infinitely often.

Writing

$$|X_n| = \left| \frac{S_n - S_{n-1}}{n} \right| \leq \left| \frac{S_n}{n} \right| + \left| \frac{S_{n-1}}{n} \right| \leq \left| \frac{S_n}{n} \right| + \left| \frac{S_{n-1}}{n-1} \right|$$

Thus, if $\left| \frac{S_n}{n} \right| > \varepsilon$ occurs only finitely many times, it is impossible that $|X_n| \to 1$ i.o. This contradiction refutes the assumption $\frac{S_n}{n} \to 0$ made above.

Remark: Does this result not contradict SLLN? No, because for SLLN, for the sum of independent, not identically distributed RV's, it is needed to hold, $\text{Var}(X_i)$ have to be uniformly bounded for all $i = 1, 2, \ldots$. In this problem, $\text{Var}(X_i) \to \infty$. 
Problem 2(a)

\[
P(X \geq a) = \int P(dw) = \int P(\sum_{\omega: X(\omega) > a} \frac{\varphi(X(\omega))}{\varphi(a)} P(dw) = \int \frac{\varphi(X(\omega))}{\varphi(a)} P(dw) = \mathbb{E} \left( \frac{\varphi(X)}{\varphi(a)} \right)
\]

In claiming we use the assumptions that \( \varphi \) is positive-valued, nondecreasing on \( \mathbb{R}^+ \).

(b) Take \( \varphi(x) = x^2 \), then \( \mathbb{E} X^2 = \text{Var}(X) \) since \( \mathbb{E} X = 0 \). Using (a), we obtain

We have \( P(|X| > a) = P(X^2 > a^2) \)

Let \( \varphi(x) = x \), then

\[
P(X^2 > a^2) \leq \frac{\mathbb{E} X^2}{a^2} = \frac{\text{Var}(X)}{a^2} \quad \text{(in the last step we rely on the assumption \( \mathbb{E} X = 0 \))}
\]

Now take \( \varphi(x) = x + \sigma^2 \), then

\[
P(X^2 > a) \leq \frac{\mathbb{E} (X^2 + \sigma^2)}{a^2 + \sigma^2} = \frac{2\sigma^2}{a^2 + \sigma^2}
\]

(c) Take \( Y = \{X \geq t\} \). By Cauchy-Schwarz,

\[
\mathbb{E} (XY)^2 \leq \mathbb{E} X^2 \mathbb{E} Y
\]

\[
= \mathbb{E} X^2 P(X \geq t)
\]

Further, \( \mathbb{E} X \leq \int_{\{X > t\}} X dP(w) + \int_{\{X \leq t\}} t dP(w) = \int_{\{X > t\}} X dP(w) + t \cdot \mathbb{P}(X \leq t) \)

\[
\therefore \mathbb{E} X - t \leq \int_{\{X > t\}} X dP(w)
\]

\( \{w: X \geq t\} \) (\( \star \))
Problem 2(c), continued:

Use this in the Cauchy-Schwarz inequality:

\[
EX^2 \cdot P(X \geq t) \geq (E(X \cdot 1_{\{X \geq t\}}))^2 = \left( \int_{\omega: X(\omega) \geq t} X(\omega) P(d\omega) \right)^2 (EX-t)^2
\]

From this we obtain

\[
P(X \geq t) \geq \frac{(EX-t)^2}{E(X^2)}
\]
Problem 2.13:

(a) Let $Z_N = \max \{ S_n, 0 \leq n \leq N \}$

It is known that $Z_N$ is a well-defined RV (as a maximum of a finite number of RVs).
Further, $Z_N$ converge pointwise to $Z$, and a pointwise limit of RVs is itself an RV.

(Note that the above claim does not follow from the fact alone that $P(Z < +\infty) = 1$; indeed, the probability $P(Z < +\infty)$ is not defined before we have proved that $Z$ is an RV)

Now let $S_n = X_1 + \ldots + X_n$; by SLLN

$$\frac{S_n}{n} \xrightarrow{a.s.} -1$$

Thus, on an event $A$ with $P(A) = 1$ for any $\varepsilon > 0$ and sufficiently large $n$,

$$\left| \frac{S_n(\omega)}{n} + 1 \right| < \varepsilon$$

in particular,

$$S_n(\omega) < n(\varepsilon - 1) \text{ a.s.}$$

i.e., $S_n(\omega) \to -\infty$ a.s.

This implies that $\sup_{n > 0} S_n < \infty$ a.s., i.e., $P(Z < +\infty) = 1$

(b) We are given that $Z_1 = \max(0, S_1)$. It is possible to construct prob. of some positive value $S_1$ with large expectation; i.e., let $S_1 \sim \text{Bernoulli}(p)$

s.t. $P(S_1 = x_1) = p$; $P(S_1 = x_2) = 1 - p$ and $E S_1 = -1$. Indeed,
Problem 2.13 part (b), continued:

Indeed, let \( x_1 < 0, x_2 > 0 \), and

\[ p(x_1) + (1-p)x_2 = -1 \]

\[ x_2 = \frac{px_1 - 1}{1-p} \]

Then \( \max(0, x_1) \) is an RV supported on \( \{0, x_2\} \), and

\[ E[Z] \geq (1-p)x_2 = px_1 - 1 \]

For a fixed \( p \), e.g., \( p = 0.1 \), it is possible to choose \( x_1 \) so that \( px_1 - 1 > L \) for any given \( L \):

\[ x_1 > \frac{1}{p} (L + 1) \]

Thus, \( E[Z] \) can be made arbitrarily large, and the answer to this question is negative.

Problem 2.17

(a) First, let us compute

\[ \ln X_n = \frac{1}{n} \sum_{i=1}^{n} \ln U_i \]

We have

\[ E[\ln U_i] = \frac{1}{2} \int_0^1 \ln x \, dx = \ln 2 - 1 \]

\[ E[\ln X_n] = n(\ln 2 - 1) \]

Thus,

\[ \frac{1}{n} \ln X_n \xrightarrow{a.s.} \ln 2 - 1 \] by SLLN.

This means that, on an event of probability 1,

\[ \left| \frac{1}{n} \ln X_n - (\ln 2 - 1) \right| \leq \epsilon \]

\[ \exp(n(-\epsilon + \ln 2-1)) \leq X_n \leq \exp(n(\epsilon + (\ln 2-1))) \]

Since both exponents are negative, this implies that

\[ P(\omega: X_n(\omega) \to 0) = 1 \]

i.e., \( X_n(\omega) \xrightarrow{a.s.} 0 \).
At the same time, assuming that \( X_n \xrightarrow{m.s.} 0 \), we run into a contradiction:

\[
E X_n^2 = (E U_n^2)^n = \left( \frac{4}{3} \right)^n
\]

i.e., \( E(X_n - 0)^2 \to \infty \)

(and there cannot be another limit by the uniqueness of limits theorem).

Thus, \( X_n \to 0 \) in p., a.s., and distribution.

(b) We have

\[
Y_n = \frac{\ln X_n}{n} \to 0
\]

i.e., \( \Theta = -1 \).

**Problem 2.19** The problem statement clearly calls for an application of the CLT. Let \( X_i \) be the random outcome of \( i \)th round.

We have

\[
E X_i = 0.4 \cdot 1 + 0.5 \cdot (-1) = -0.1
\]

\[\text{Var}(X_i) = 0.89\]

Let \( S_n = X_1 + \ldots + X_n \) (for us \( n = 100 \))

\[
\frac{S_n - n \mu}{\sigma \sqrt{n}} \xrightarrow{d} N(0,1)
\]

We need to find \( P(S_n > 0) = P\left( \frac{S_n + 10}{\sqrt{89}} > \frac{10}{\sqrt{89}} \right) = 1 - P(X < \frac{10}{\sqrt{89}}) \)

\[= 1 - \Phi \left( \frac{10}{\sqrt{89}} \right) \approx 0.15\]
Problem 2.27

(a) A typical realization is like this:

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad n \]

(b) Compute the first few values: \( E_X, E_{X^2}, E_{X^3}, E_{X^4} \)

\[ E_{X^2} = \frac{1}{2}; \quad E_{X^3} = \frac{3}{4}; \quad E_{X^4} = \frac{5}{8} \]

and form a conjecture that

\[ E_{X^n} = \sum_{i=0}^{n-1} (-\frac{1}{2})^i, \quad n \geq 2 \]

A proof by induction is straightforward.

(c) First prove that \( D_{n+1} \leq D_n \) a.s. For this, check that

\[ D_{n+1} = U_n D_n \] (this is a direct computation), and thus

\[ |X_{n+1} - X_n| = D_{n+1} \leq D_n \text{ a.s.} \]

Further, \( D_n \xrightarrow{a.s.} 0 \), and thus \( |X_{n+1} - X_n| \xrightarrow{a.s.} 0 \),

which implies that \( X_n \) converges in the a.s. sense by the Cauchy criterion.