Problem 1. Let $X$ be an exponential RV, $X \sim \text{Exp}(\lambda)$ and let $x_0 > 0$.

(a) Show that $P(X > s + t | X > t) = P(X > s)$.

(b) Show that $E(X | X > x_0) = x_0 + E X$, $\text{Var}(X | X > x_0) = \text{Var}(X)$.

(c) Let $X_1 \sim \text{Exp}(\lambda_1)$, $X_2 \sim \text{Exp}(\lambda_2)$ be independent RVs. Show that $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.

(d) Let $X_i \sim \text{Exp}(\lambda_i), i = 1, 2, \ldots, n$ be independent RVs. Show that $Y \triangleq \min(X_1, \ldots, X_n)$ is an exponential RV $Y \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n)$.

(e) Continuing from part (d), show that $P(X_1 = \min_{1 \leq i \leq n} X_i) = \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}$.

Problem 2. Consider a discrete-time Markov chain with the state space $S = 0, 1, 2, \ldots$ and transition probabilities

$\begin{align*}
p_{0,1} &= 1, \\
p_{i,i+1} &= a_i = 1 - p_{i,0}, i = 1, 2, \ldots.
\end{align*}$

We assume that $a_i > 0$ for all $i$.

(a) Show that all the states are recurrent if and only if $\prod_{i=1}^\infty a_i = 0$.

(b) Suppose that all the states are recurrent. Show that they are positive recurrent if and only if

$$\sum_{n=1}^\infty \prod_{i=1}^n a_i < \infty.$$ 

Problem 3. Let $N(t), t \geq 0$ be a Poisson process with rate $\lambda > 0$. Define the process $X(t)$ by

$$X(t) = \frac{N(t + \delta^2) - N(t)}{\delta}, \quad t \geq 0,$$

where $\delta > 0$.

(a) Is $X(t)$ a Poisson process?

(b) Find the mean $E(X(t))$.

(c) Find the autocovariance function $C_X(t_1, t_2)$ of the process $X(t)$ for $t_1 = 1, t_2 = 2$, and $\delta = 1$.

Problem 4. Consider a (continuous-time) Markov chain with states $S = \{0, 1, \ldots, N\}$ given by the generator matrix $Q$ of the form

$$Q = \begin{pmatrix}
-\lambda & \lambda & 0 & \ldots & 0 & 0 \\
0 & -\lambda & \lambda & \ldots & 0 & 0 \\
0 & 0 & -\lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & -\lambda & \lambda \\
0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}$$

(a) Write the forward equations of the chain.

(b) Solve the forward equations and find the matrix $P(t)$ (note that $p_{ij} = 0$ for $i > j$).

(c) Compute $\lim_{t \to r^+} p_{i,j}(t)$ for all $0 \leq i < j \leq N$. Conclude that the states $0, 1, \ldots, N - 1$ are transient, and state $N$ is absorbing.

(d) Sketch sample paths of this chain.