Quasiperiodically Forced Damped Pendula and Schrödinger Equations with Quasiperiodic Potentials: Implications of Their Equivalence

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(Received 28 June 1985)

Certain first-order nonlinear ordinary differential equations exemplified by strongly damped, quasiperiodically driven pendula and Josephson junctions are isomorphic to Schrödinger equations with quasiperiodic potentials. The implications of this equivalence are discussed. In particular, it is shown that the transition to Anderson localization in the Schrödinger problem corresponds to the occurrence of a novel type of strange attractor in the pendulum problem. This transition should be experimentally observable in the frequency spectrum of the pendulum or Josephson junction.

PACS numbers: 05.45.+b, 03.20.+i

The strongly damped, driven pendulum with time-dependent forcing and/or string length is described by

$$\nu\, d\phi/dt=\gamma(t)\cos\phi+f(t), \tag{1}$$

where ν, γ, and f represent frictional, gravitational, and external torques, respectively. In (1) we have neglected the inertial term, $d^2\phi/dt^2$, as is appropriate in the case of strong damping and suitably slow time dependence of γ and f. Equation (1) also describes Josephson junctions with a large shunt conductance. We take $\gamma(t)$ and $f(t)$ to be quasiperiodic with two incommensurate frequencies, i.e., $f(t)=\hat{f}(\omega_1 t,\omega_2 t)$, where $\hat{f}$ is 2π periodic in both of its arguments, and $\omega_1/\omega_2=\omega$ is irrational. Nonlinear differential equations of second order with *periodic* forcing have been extensively studied[1] and are known to exhibit interesting nonlinear dynamical behavior (e.g., period doubling, intermittency, etc.). *Quasiperiodic* forcing, however, has received little attention.[2,3] As we shall show, quasiperiodic forcing, *even in the case of a single first-order differential equation,* can lead to interesting dynamics. In particular, as a forcing parameter is increased the solutions undergo a transition to a novel type of strange attractor with distinct power-spectrum characteristics.

Our analysis of Eq. (1) will be based on a transformation originally used by Prüfer in 1926 to study the spectrum of the Schrödinger equation,

$$-d^2\Psi/dx^2+\lambda V(x)\Psi(x)=E\Psi(x). \tag{2}$$

Here we consider (2) in the case of a quasiperiodic potential $V(x)=v(k_1x,k_2x)$, where v is 2π periodic in its arguments. Schrödinger equations with quasiperiodic potentials have been the subject of much recent study (see reviews by Souillard[4] and Simon[5]) because of applications in condensed matter physics and also because they are an intermediate case interpolating between purely periodic potentials, for which the states are extended (Bloch waves), and random potentials, for which the states are Anderson localized.[4,6] Past work has shown that, for k_1/k_2 a typical irrational number[7] and for small λ, the states are typically extended, while for large λ localized states are typical. In both cases allowed energies fall on a Cantor set of finite measure with stop bands on the complement of this Cantor set.[8]

We now use Prüfer's transformation[9] to show the connection between Eqs. (1) and (2). For the complex amplitudes, $a(x)=\Psi'+ig\Psi$ and $b(x)=\Psi'-ig\Psi$, Eq. (2) yields $a'=i(\alpha a-\beta b)$ and $b'=i(\beta a-\alpha b)$, where $2\alpha=(k^2/g)+g$, $2\beta=(k^2/g)-g$, g is an arbitrary constant, $k^2(x)=E-\lambda V(x)$, and the prime denotes differentiation with respect to x. For Ψ real, $|a|=|b|$ and $a/b=\exp[i\phi(x)]$ with ϕ real. We then obtain

$$\begin{aligned} d\phi/dx&=g^{-1}\{[g^2+k^2(x)]+[g^2-k^2(x)]\cos\phi\}\\ &=\hat{h}(\phi,x), \end{aligned} \tag{3}$$

which is of the same form as (1). By a change of independent variables an arbitrary positive function can be introduced multiplying the right-hand side of (3). [To transform (2) into (1) with a positive γ, we need to take $g^2>\max k^2(x)$.]

For the Schrödinger equation the integrated density of states (the number of states per unit length with energies less than E) is equal to[9] $1/2\pi$ times the *winding number* $\hat{\kappa}(E)=\langle\hat{h}(\phi(x),x)\rangle_x$, where $\langle\ldots\rangle_x$ denotes an average over x. The *Lyapunov exponent* for (3) is $\hat{\Lambda}=\langle\partial\hat{h}/\partial\phi\rangle_x$ and also has the significance of being the negative of the exponentiation rate of $|a|^2$. For (E,λ) in the localized and stop-band regions, $\hat{\Lambda}$ computed from (3) with typical initial conditions is *negative.* For extended states $\hat{\Lambda}=0$.

We now consider the pendulum and Josephson-junction equation, (1), with $\nu=1$, $\gamma(t)=1$, and quasiperiodic forcing $f(t)=K_0+V_0(\cos\omega_1 t+\cos\omega_2 t)$, where $\omega_2=1$ and $\omega_1=\omega=(\sqrt{5}-1)/2$ (the reciprocal of the golden mean). This corresponds to

$$k^2(x(t))=g^2[f(t)-1]/[f(t)+1]$$

with the change of variables

$$x(t)=(2g)^{-1}[(K_0+1)t+V_0(\sin t+\omega^{-1}\sin\omega t)],$$

which is valid when $K_0>2V_0-1$. With a switch to the t variable, the winding number κ and Lyapunov exponent Λ for (1) are, to within a constant factor, the same as those for (3), $\kappa/\hat{\kappa}=\Lambda/\hat{\Lambda}=(K_0+1)/2g$. Figure 1 shows a numerically obtained map of the K_0-V_0 plane giving regions where Λ is negative (hatched) or zero. Figure 2 shows Λ and κ vs K_0 at the value of V_0 indicated by the arrow in Fig. 1. Making use of the analogy with Eq. (2) we can make the following statements concerning Eq. (1). First κ vs K_0 is a "devil's staircase": a continuous, nondecreasing curve with a dense set of open intervals on which κ is constant and given by[9]

$$\kappa = m\omega_1 + l\omega_2, \tag{4}$$

with l and m integers. Between these plateaus is a Cantor set, generally of finite measure, on which κ increases with K_0. For low V_0 or large K_0 [small λ or large E in (2)], Λ is zero on the Cantor set and negative on the plateaus. For large V_0, Λ becomes negative on the Cantor set as well as on the plateaus. In the Schrödinger problem, the plateaus are the stop bands, the Cantor set with $\Lambda=0$ corresponds to the extended regime, and the Cantor set with $\Lambda<0$ corresponds to the localized regime.

In Fig. 1 the regions where (4) holds appear as narrow tongues emerging at small V_0. The set of points where the tongues touch each other appear to lie on a smooth (but nonunique) critical curve which separates the localized and extended regimes. A crude approximation to such a critical curve is indicated in Fig. 1.

We now consider the surface of section for (1) by strobing the system at times $t_j=(2j\pi/\omega_2)+t_0$ (j is an integer) and plot $\phi_j=\phi(t_j)(\mathrm{mod}2\pi)$ versus $\theta_j=\omega t_j(\mathrm{mod}2\pi)$. The solution of the differential equation thus generates a discrete time map, $\phi_{j+1}=M(\phi_j,\theta_j)$, $\theta_{j+1}=(\theta_j+2\pi\omega)(\mathrm{mod}2\pi)$, where M is invertible for ϕ. In the case of Kronig-Penny models, where V consists of δ functions (and, hence,[5] also the *discrete* Schrödinger equation), it is simple to compute M explicitly from (3).

The extended, stop-band, and localized solutions of the Schrödinger equation give rise to surface-of-section plots with qualitatively different characteristics. For the extended case typically the orbit in the surface of section densely fills the two-torus (θ,ϕ), apparently generating a smooth density of points, the winding number κ is irrationally related to ω_1 and ω_2, and thus a 2π-periodic function of $\phi(t)$ will possess frequency components at $l\omega_1+m\omega_2+n\kappa$ (l,m,n are integers). Thus extended states of (2) correspond to three-frequency quasiperiodic orbits of (1).

In a stop band, the attracting orbit in the surface of section lies on a smooth single-valued curve, $\phi=F(\theta)$, which wraps m times around the torus in ϕ for each time that it wraps once around in θ. [This is a consequence of (4).] Since κ satisfies Eq. (4), a 2π-periodic function of ϕ will have frequency components at $l\omega_1+m\omega_2$. Thus stop bands of (2) correspond to two-frequency quasiperiodic attractors of Eq. (1). The attracting curve $\phi=F(\theta)$ is invariant under the map. In addition, there is also an unstable invariant curve, $\phi=\tilde{F}(\theta)$, which repells initial conditions (attracts them as $t\to-\infty$).

To see whether $\phi=F(\theta)$ also applies to the "localized" case, we initialized a large number of points at a single initial θ value, but with different initial ϕ values. After a large number of iterates, we find that the orbits are all attracted to a single value ϕ_j. This implies that the attractor for the localized case also satisfies a functional relationship, $\phi=F(\theta)$. However, from the correspondence with localized states of (2),

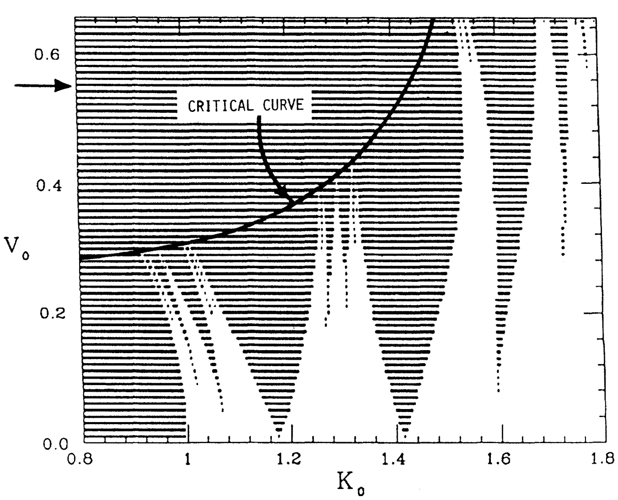


FIG. 1. Plot K_0-V_0 plane. The hatched regions indicate a negative Lyapunov exponent for Eq. (1).

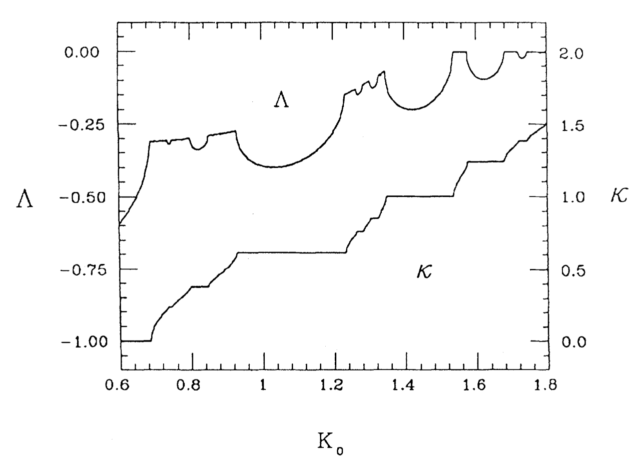


FIG. 2. Lyapunov exponent and winding number for Eq. (1) as a function of K_0 for $V_0=0.55$, indicated by an arrow in Fig. 1.

the rotation number is typically irrationally related to ω_1 and ω_2. Thus $F(\theta)$ cannot be a continuous curve. Otherwise it would continuously join up with itself after m wraps in ϕ implying $\kappa = m\omega_1 + n\omega_2$. Hence $F(\theta)$ must contain discontinuities. If $F(\theta)$ is discontinuous at some $\theta = \theta_d$, then it must be discontinuous everywhere, since the θ map is ergodic for irrational ω. This is also suggested by Fig. 3(a) and from its similarity with the analyzable case discussed in Ref. 3. Hence this attractor is *geometrically strange*,[3] but, in contrast with strange attractors usually encountered in nonlinear dynamics, it is *not* chaotic (the Lyapunov exponent Λ is negative). Thus, *in the localized regime for (2), Eq. (1) has strange nonchaotic attractors with* $\Lambda < 0$. The possibility of strange nonchaotic attractors with $\Lambda < 0$ and positive measure in parameter space has been discussed for systems forced at two incommensurate frequencies in Ref. 3.

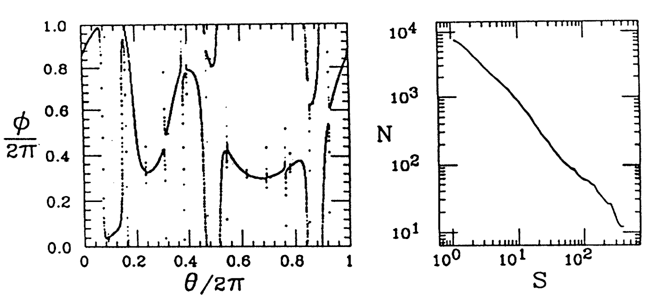
Approaching the edge of a region where (4) holds, the transition from two- to three-frequency motion (below the critical curve in Fig. 1) occurs when the stable and unstable curves coalesce[2] [i.e., $F(\theta) = \tilde{F}(\theta)$ at the transition and the curves remain smooth]. In the transition from two-frequency to strange solutions, however, the stable and unstable invariant curves become more and more wiggly as one nears the edge of a two-frequency band,[2] and the curves approach each other only at certain points, appearing to touch at the transition. An indication that they must touch for the localized regime[10] is the fact that for a localized solution, $|a|$ decays at both $x \to +\infty$ and $x \to -\infty$, and that decay at $x \to -\infty$ corresponds to an orbit of (1) on the stable invariant curve, while decay at $x - +\infty$ corresponds to the orbit being on the unstable invariant curve.

Now consider the discrete-time Fourier transform of a sequence $P(\phi_j)$, where P is some smooth 2π-periodic function. For strange attractors and for two-frequency quasiperiodicity, $\phi = F(\theta)$, and we define $C(\theta) = P(F(\theta))$. Expanding $C(\theta)$ as $\sum c(m)\exp(im \times\theta)$, and noting that $\theta_j = (2\pi j\omega + \theta_0)(\mathrm{mod}2\pi)$, we see that the discrete-time Fourier transform of $C(\theta_j)$ is $\tilde{C}(\Omega) = \sum c(m)\delta(\Omega - 2\pi[m\omega(\mathrm{mod}1)])$, where the frequency variable Ω is restricted to the range $2\pi > \Omega \geq 0$. Thus $\tilde{C}(\Omega)$ consists of discrete peaks at $2\pi[m\omega(\mathrm{mod}1)]$ with strengths $|c(m)|$. For strange attractors the discontinuous nature of $F(\theta)$ and $C(\theta)$ results in a rich high-harmonic content of $c(m)$, and one might expect a power law $|c(m)| \sim m^{-1/\alpha}$, for large m. With $N(S)$ the number of spectral components larger than some value of S, the power-law spectrum implies

$$N(S) \sim S^{-\alpha}, \tag{5}$$

for strange attractors. Figure 3(b) shows results for $N(S)$ with $P(\phi) = \cos\phi$ obtained by Fourier transformation of the strobed numerical solution[11] of (1) for a strange-attractor case. The result confirms (5) [$\alpha \cong 1.2$ for Fig. 3(b)]. Numerically we find that α falls in the range $1 < \alpha < 2$. Following a fixed irrational winding number κ, as V_0 is increased, a three-frequency quasiperiodic orbit will undergo a transition to a strange attractor at some critical value $V_0 = V_c(\kappa)$. We find that α is near 2 at V_c and rapidly decreases toward 1 as V_0 is raised above V_c. The exponent $\alpha = 1$ can be understood by the following crude argument. The discontinuities of $F(\theta)$ appear to result from successive iterates of jumps of ϕ, by approximately 2π, over a short range in θ [see Fig. 3(a)]. In the strongly localized regime the width $\delta\theta$ of a jump decreases sharply on iteration of the map. For $\cos\phi$ these jumps appear as spikes of height $\sim O(1)$ and width $\sim \delta\theta$. Thus one jump gives rise to Fourier components $|c(m)| \sim \delta\theta$ for $|m| \lesssim 1/\delta\theta$. Summing over jumps, we expect the scaling $|c(m)| \sim m^{-1}$. We believe that the power-law behavior of $N(S)$ is an important signature that should make such attractors experimentally identifiable. Finally we note that for the two- and three-frequency quasiperiodic cases[10] $N \sim \ln 1/S$ and $N \sim (\ln/1S)^2$, which is clearly distinguishable from (5).

All of the previous discussion has considered the first-order differential equation describing a pendulum or Josephson junction, Eq. (1). This equation is special in that it is related to the linear, time-independent Schrödinger equation by Prüfer's transformation. To what extent do equations of the more general type $d\phi/dt = h(\phi,t)$, where h is periodic in ϕ and quasiperiodic in t, exhibit behavior similar to that for Eq. (1)? To address this question we have performed a series of numerical studies[10] with the main results as follows. At low forcing the generic case is similar to that of Eq. (1) in that three-frequency quasiperiodic motions occur on a Cantor set of finite measure, while two-frequency quasiperiodic motions occur on a dense set of intervals. However, the generic case differs in



(a) (b)

FIG. 3. (a) Surface-of-section plot for a strange attractor for the pendulum equation at $K_0 = 1.34$, $V_0 = 0.55$. The system is strobed at times $t_j = (j + 3/16)2\pi$, j integer. (b) Log-log plot of $N(S)$ for the case in (a).

that the plateaus with two-frequency quasiperiodic attractors occur at winding numbers $\kappa = [(l/n)\omega_1 + (m/n)\omega_2](\mathrm{mod}1)$ (l,m,n are integers), corresponding to a n-times-multivalued $F(\theta)$ in the surface of section [cf. Eq. (4)]. Similar to Eq. (1), as the nonlinearity is raised, strange attractors occur on a Cantor set separating intervals on which two-frequency quasiperiodic attractors occur, and the frequency spectra associated with the strange attractors appear to be characterized by Eq. (5) with exponents $2 > \alpha > 1$. The main difference with Eq. (1) is that the measure of the Cantor set with strange attractors seems to be zero, or at least is very small. The latter does not, however, prevent observation, since all that is required to see spectra with the property of Eq. (5) is the tuning of a single parameter.[12]

In conclusion, the quasiperiodically driven-damped pendulum equation[13] may be said to represent the simplest continuous-time systems with strange attractors. These attractors have a clear signature in the character of their frequency spectra, and this signature should be experimentally observable.

This research was supported in part by the National Science Foundation under Grant No. PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, at the University of California at Santa Barbara. It was also supported by the Swedish Energy Research Council, and the U.S. Department of Energy.

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[5]B. Simon, Adv. Appl. Math. **3**, 463 (1982).

[6]For a localized state, $\int |\Psi|^2 dx < \infty$, and the eigenfunctions are typically exponentially decaying for large $|x|$.

[7]For a "typical" irrational k, there exist $c > 0$ and $\beta > 1$ such that $|k - p/q| > cq^{-\beta}$ for all integers p and q. The set of such irrationals has full measure.

[8]In general, the role of singular continuous spectra (Refs. 4 and 5) is not clear, but in our numerical studies (A. Bondeson, E. Ott, and T. M. Antonsen, Jr., to be published) at $\omega = (\sqrt{5} - 1)/2$ we have not seen evidence of zero-measure Cantor sets of allowed energies except on the critical curve (cf. Fig. 1).

[9]R. Johnson and J. Moser, Commun. Math. Phys. **84**, 403 (1983).

[10]Bondeson, Ott, and Antonsen, Ref. 8.

[11]Similar results are found for the continuous-time Fourier transform over a suitable finite frequency range (Ref. 10).

[12]The shape of the power spectrum varies continuously with the parameters. Thus, near the edge of a two-frequency band, in the strongly forced regime of (1), $N(S)$ *approaches* a power law.

[13]Equation (1) with $\nu = \frac{1}{2}$ and γ constant is related to the Ricatti equation of the parametric interaction problem with a time-dependent frequency mismatch, $\ddot{A} - 2if(t)\dot{A} - \gamma^2 A = 0$; see D. Anderson and A. Bondeson, Phys. Scr. **14**, 324 (1976). By a change of both independent and dependent variables, $g\,dx = (f+\gamma)dt$, $A \exp(-i\int f\,dt) = \Psi' - ig\Psi$, the parametric equation transforms into Eq. (2), where $k^2(x) = g^2(f-\gamma)/(f+\gamma)$ is real. The quasiperiodic case discussed here corresponds to a parametric interaction with a three-frequency pump.