Digital Logic Design
ENEE 244-010x

Lecture 12

Announcements

- HW5 due today
- HW6 up on course webpage, due at the beginning of class on Wednesday, 10/28.

Agenda

- Last time:
  - Quine-McClusky (4.8)
  - Petrick's Method (4.9)
  - Table Reductions (4.10)

- This time:
  - Multiple Output Simplification Problem (4.12, 4.13)

The Multiple-Output Simplification Problem

- General combinational networks can have several output terminals.
- The output behavior of the network is described by a set of functions $f_1, f_2, \ldots, f_m$, one for each output terminal, each involving the same input variables, $x_1, x_2, \ldots, x_n$.
- The set of functions is represented by a truth table with $m + n$ columns.
- Objective is to design a multiple-output network of minimal cost.
- Formally: A set of normal expressions that has associated with it a minimal cost as given by some cost criteria.
- Cost criteria: number of gates or number of gate inputs in the realization.

Pitfalls of Naïve Approach

- Multiple-output minimization problem is normally more difficult than sharing common terms in independently obtained minimal expressions.
- Consider:

$$f_1(x, y, z) = \sum m(1,3,5)$$

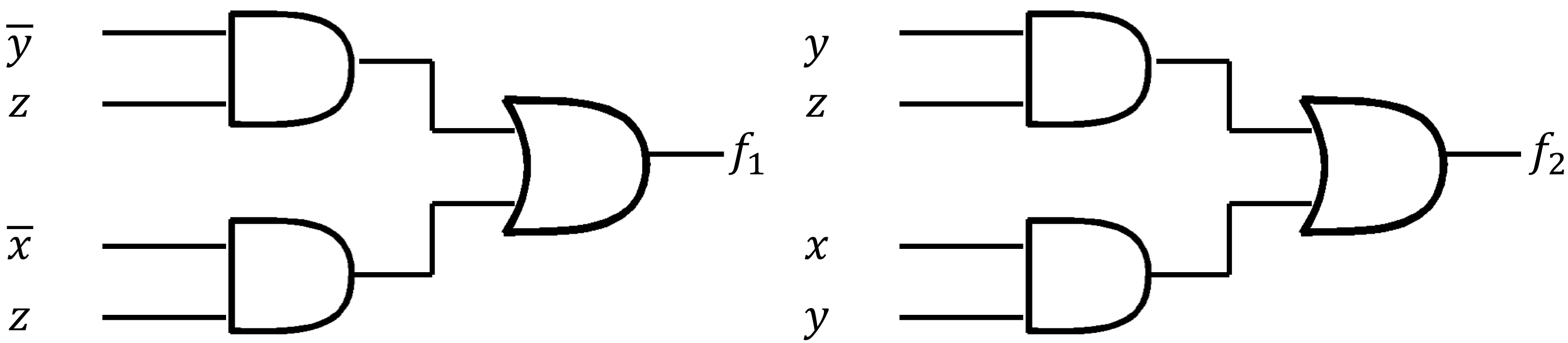
$$f_2(x, y, z) = \sum m(3,6,7)$$

$$f_1(x, y, z) = \overline{y}z + \overline{x}z$$

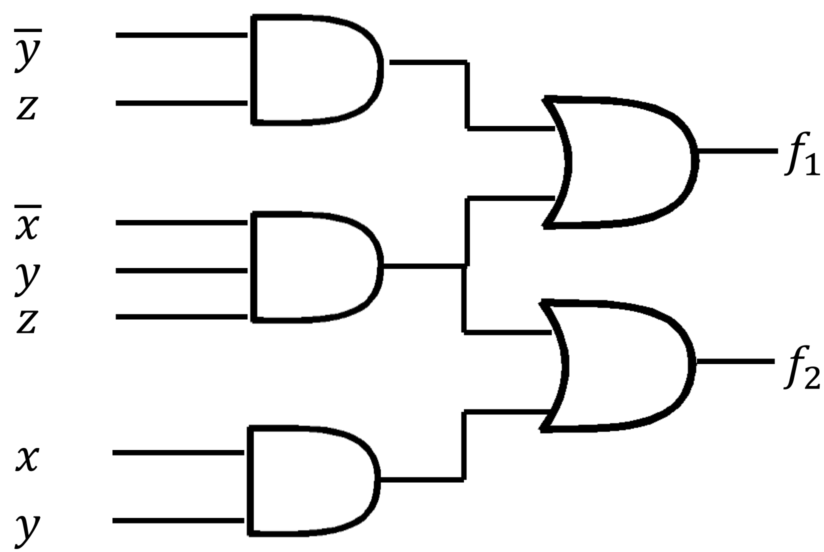
$$f_2(x, y, z) = yz + xy$$

Pitfalls of Naïve Approach

Naïve Approach:



Better Approach:



Multiple Output Prime Implicants

- A multiple-output prime implicant for a set of Boolean functions $f_1, f_2, \ldots, f_m$, is a product term that:
  - Is a prime implicant of one of the individual functions
  - Is a prime implicant of one of the product functions $\Pi_{i \in S} f_i,\ S \subseteq \{1, .., m\}$

Examples of Multiple Output Prime Implicants

x	y	z	f_1	f_2
0	0	0	1	0
0	0	1	0	1
0	1	0	1	1
0	1	1	0	0
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1

$y\overline{z}$ is a prime implicant of f_1

Examples of Multiple Output Prime Implicants

x	y	z	f_1	f_2	$f_1 \cdot f_2$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	1	1
0	1	1	0	0	0
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	0	1	0

$\overline{x}y\overline{z}$ is a prime implicant of $f_1 \cdot f_2$

Multiple Output Prime Implicants

Theorem: Formulas that achieve the multiple-output minimal sum consist only of sums of multiple-output prime implicants.

Tagged Product Terms

- Term consists of two parts: a kernel and a tag.
  - Kernel: product term involving the variables of the function
  - Tag: Appended to denote which functions are implied by its kernel.

Tagged Product Term Example

x	y	z	f_1	f_2
0	0	0	1	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	1
1	0	0	0	0
1	0	1	1	--
1	1	0	0	0
1	1	1	1	1

Algebraic Form	Binary Form
$\overline{x}\,\overline{y}\,\overline{z}f_1 -$	$000f_1 -$
$\overline{x}\,\overline{y}zf_1f_2$	$001f_1f_2$
$\overline{x}y\overline{z}f_1f_2$	$010f_1f_2$
$\overline{x}yz - f_2$	$011 - f_2$
$x\overline{y}zf_1f_2$	$101f_1f_2$
$xyzf_1f_2$	$111f_1f_2$

Quine-McClusky for tagged multiple-output prime implicants

0	0	0	0	f_1	–	✓
1	0	0	1	f_1	f_2	✓
2	0	1	0	f_1	f_2	
3	0	1	1	–	f_2	✓
5	1	0	1	f_1	f_2	✓
7	1	1	1	f_1	f_2	✓

(0,1)	0	0	–	f_1	–	
(0,2)	0	–	0	f_1	–	
(1,3)	0	–	1	–	f_2	✓
(1,5)	–	0	1	f_1	f_2	
(2,3)	0	1	–	–	f_2	
(3,7)	–	1	1	–	f_2	✓
(5,7)	1	–	1	f_1	f_2	

(1,3,5,7)	–	–	1	–	f_2
(1,5,3,7)	–	–	1	–	f_2

Why doesn't (0,1,2,3) appear?

- The tag of a generated term has f_i iff f_i appears in both the tags of the generating terms.
- A generating term is checked only if its tag is identical to the tag of the generated term.

Minimal Sums Using Petrick's Method

Multiple Outputs Prime Implicant Tables

			m_0	m_1	m_2	m_5	m_7	m_1	m_2	m_3	m_7
f_1	B	$\overline{x}\,\overline{y}$	X	X							
f_1	C	$\overline{x}\,\overline{z}$	X		X						
f_2	A	z						X		X	X
f_2	E	$\overline{x}z$							X	X	
$f_1 \cdot f_2$	D	$\overline{y}z$		X		X		X			
$f_1 \cdot f_2$	F	xz				X	X				X
$f_1 \cdot f_2$	G	$\overline{x}y\overline{z}$			X				X		

Multiple Outputs Prime Implicant Tables

			m_0	m_1	m_2	m_5	m_7	m_1	m_2	m_3	m_7
f_1	B	$\overline{x}\,\overline{y}$	X	X							
f_1	C	$\overline{x}\,\overline{z}$	X		X						
f_2	A	z						X		X	X
f_2	E	$\overline{x}z$							X	X	
$f_1 \cdot f_2$	D	$\overline{y}z$		X		X		X			
$f_1 \cdot f_2$	F	xz				X	X				X
$f_1 \cdot f_2$	G	$\overline{x}y\overline{z}$			X				X		

When writing down p-expression, must make a distinction between primes associated with different functions.

Multiple Outputs Prime Implicant Tables

			m_0	m_1	m_2	m_5	m_7	m_1	m_2	m_3	m_7
f_1	B	$\overline{x}\,\overline{y}$	X	X							
f_1	C	$\overline{x}\,\overline{z}$	X		X						
f_2	A	z						X		X	X
f_2	E	$\overline{x}z$							X	X	
$f_1 \cdot f_2$	D	$\overline{y}z$		X		X		X			
$f_1 \cdot f_2$	F	xz				X	X				X
$f_1 \cdot f_2$	G	$\overline{x}y\overline{z}$			X				X		

P-expression: $(B_1 + C_1)(B_1 + D_1)(C_1 + G_1)(D_1 + F_1)F_1(A_2 + D_2)(E_2 + G_2)(A_2 + E_2)(A_2 + F_2)$

Manipulating P-expression into sum of product form

$$p = (B_1 + C_1)(B_1 + D_1)(C_1 + G_1)(D_1 + F_1)F_1(A_2 + D_2)(E_2 + G_2)(A_2 + E_2)(A_2 + F_2)$$
$$= A_2B_1C_1E_2F_1 + A_2B_1C_1F_1G_2 + B_1C_1D_2E_2F_1F_2 + A_2B_1E_2F_1G_1 + A_2B_1F_1G_1G_2 + B_1D_2\mathrm{E}_2F_1F_2G_1 + A_2C_1D_1E_2F_1 + A_2C_1D_1F_1G_2 + C_1D_1D_2E_2F_1F_2$$

When calculating cost of a product term, we can disregard subscripts.

i.e. F_1F_2 is the same cost as F.

Calculating Cost of Product Terms

- The term $A_2B_1F_1G_1G_2$ yields
  - $f_1(x,y,z) = \overline{x}\,\overline{y} + xz + \overline{x}y\,\overline{z}$
  - $f_2(x,y,z) = z + \overline{x}y\overline{z}$
- The term $C_1D_1D_2E_2F_1F_2$ yields
  - $f_1(x,y,z) = \overline{x}\,\overline{z} + \overline{y}z + xz$
  - $f_2(x,y,z) = \overline{y}z + \overline{x}y + xz$

Calculating Cost of multiple output combinational network

$$f_1, \dots, f_m$$

$$\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{p} \beta_j$$

Where $t_1, \dots, t_p$ is the set of distinct terms, β_j is equal to the number of literals in t_j, unless the term consists of a single literal, in which case $\beta_j = 0$. Let α_i be the number of terms in f_i unless there is only a single term, in which case $\alpha_i = 0$.

Calculating Cost of Product Terms

- The term $A_2B_1F_1G_1G_2$ yields
  - $f_1(x,y,z) = \overline{x}\,\overline{y} + xz + \overline{x}y\,\overline{z}$
  - $f_2(x,y,z) = z + \overline{x}y\overline{z}$

  Distinct terms: $\overline{x}\,\overline{y}, xz, \overline{x}y\,\overline{z}, \mathrm{z}$
  Beta costs: 2 + 2 + 3 + 0 = 7
  Alpha costs = 3 + 2 = 5
  Total cost: 12
- The term $C_1D_1D_2E_2F_1F_2$ yields
  - $f_1(x,y,z) = \overline{x}\,\overline{z} + \overline{y}z + xz$
  - $f_2(x,y,z) = \overline{y}z + \overline{x}y + xz$

  Distinct terms: $\overline{x}\,\overline{z}, \overline{y}z, xz, \overline{x}y$
  Beta costs: 2+2+2+2 = 8
  Alpha costs = 3 +3 = 6
  Total cost: 14

Minimal Sums using Table Reduction

Table Reduction

		m_0	m_1	m_2	m_5	m_7	m_1	m_2	m_3	m_7	Cost
B	$\bar{x}\,\bar{y}$	X	X								3
C	$\bar{x}\,\bar{z}$	X		X							3
A	z						X		X	X	1
E	$\bar{x}y$							X	X		3
D	$\bar{y}z$		X		X		X				3,4
F	xz				X	X				X	3,4
G	$\bar{x}y\bar{z}$			X				X			4,5

Cost in case D is used in f_1 or f_2 but not both, Cost in case D is used in both f_1, f_2

Table Reduction

		m_0	m_1	m_2	m_5	m_7	m_1	m_2	m_3	m_7	Cost
B	$\overline{x}\,\overline{y}$	X	X								3
C	$\overline{x}\,\overline{z}$	X		X							3
A	z						X		X	X	1
E	$\overline{x}y$							X	X		3
D	$\overline{y}z$		X		X		X				3,4
F	xz				X	X				X	3,4
G	$\overline{x}y\overline{z}$			X				X			4,5

Essential prime implicant for f_1.

Table Reduction

		m_0	m_1	m_2	m_1	m_2	m_3	m_7	Cost
B	$\overline{x}\,\overline{y}$	X	X						3
C	$\overline{x}\,\overline{z}$	X		X					3
A	z				X		X	X	1
E	$\overline{x}y$					X	X		3
D	$\overline{y}z$		X		X				3,4
*1 F	xz							X	1
G	$\overline{x}y\overline{z}$			X		X			4,5

m_7 column cannot be removed from f_2 part since xz is not essential for f_2.

$$f_1 = xz + \cdots$$
$$f_2 = \cdots$$

Table Reduction

		m_0	m_1	m_2	m_1	m_2	m_3	m_7	Cost
B	$\bar{x}\,\bar{y}$	X	X						3
C	$\bar{x}\,\bar{z}$	X		X					3
A	z				X		X	X	1
E	$\bar{x}y$					X	X		3
D	$\bar{y}z$		X		X				3,4
*1 F	xz							X	1
G	$\bar{x}y\bar{z}$			X		X			4,5

Dominated Rows

$$f_1 = xz + \cdots$$
$$f_2 = \cdots$$

Table Reduction

		m_0	m_1	m_2	m_1	m_2	m_3	m_7	Cost
B	$\bar{x}\,\bar{y}$	X	X						3
C	$\bar{x}\,\bar{z}$	X		X					3
A	z				X		X	X	1
E	$\bar{x}y$					X	X		3
D	$\bar{y}z$		X		X				3,4
*1 F	xz							X	1
G	$\bar{x}y\bar{z}$			X		X			4,5

Dominated Rows
Row A dominates Row F
Cost for Row A is not greater than cost for Row F.

$$f_1 = xz + \cdots$$
$$f_2 = \cdots$$

Table Reduction

		m_0	m_1	m_2	m_1	m_2	m_3	m_7	Cost
B	$\bar{x}\,\bar{y}$	X	X						3
C	$\bar{x}\,\bar{z}$	X		X					3
A	z				X		X	X	1
E	$\bar{x}y$					X	X		3
D	$\bar{y}z$		X		X				3,4
G	$\bar{x}y\bar{z}$			X		X			4,5

$$f_1 = xz + \cdots$$
$$f_2 = \cdots$$

Table Reduction

		m_0	m_1	m_2	m_1	m_2	m_3	m_7	Cost
B	$\bar{x}\,\bar{y}$	X	X						3
C	$\bar{x}\,\bar{z}$	X		X					3
A	z				X		X	X	1
E	$\bar{x}y$					X	X		3
D	$\bar{y}z$		X		X				3,4
G	$\bar{x}y\bar{z}$			X		X			4,5

Only row that covers m_7

$$f_1 = xz + \cdots$$
$$f_2 = \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
B	$\bar{x}\,\bar{y}$	X	X			3
C	$\bar{x}\,\bar{z}$	X		X		3
*2 A	z					1
E	$\bar{x}y$				X	3
D	$\bar{y}z$		X			3,4
G	$\bar{x}y\bar{z}$			X	X	4,5

Delete m_1, m_3, m_7

$$f_1 = xz + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
B	$\overline{x}\,\overline{y}$	X	X			3
C	$\overline{x}\,\overline{z}$	X		X		3
E	$\overline{x}y$				X	3
D	$\overline{y}z$		X			3
G	$\overline{x}y\overline{z}$			X	X	4,5

Delete row A

$$f_1 = xz + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
B	$\overline{x}\,\overline{y}$	X	X			3
C	$\overline{x}\,\overline{z}$	X		X		3
E	$\overline{x}y$				X	3
D	$\overline{y}z$		X			3
G	$\overline{x}y\overline{z}$			X	X	4,5

Row D is dominated by Row B.

$$f_1 = xz + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
B	$\overline{x}\,\overline{y}$	X	X			3
C	$\overline{x}\,\overline{z}$	X		X		3
E	$\overline{x}y$				X	3
D	$\overline{y}z$		X			3
G	$\overline{x}y\overline{z}$			X	X	4,5

Row D is dominated by Row B.

$$f_1 = xz + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
B	$\overline{x}\,\overline{y}$	X	X			3
C	$\overline{x}\,\overline{z}$	X		X		3
E	$\overline{x}y$				X	3
G	$\overline{x}y\overline{z}$			X	X	4,5

Row D is dominated by Row B.

$$f_1 = xz + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
B	$\overline{x}\,\overline{y}$	X	X			3
C	$\overline{x}\,\overline{z}$	X		X		3
E	$\overline{x}y$				X	3
G	$\overline{x}y\overline{z}$			X	X	4,5

Row B is the only row covering m_1

$$f_1 = xz + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_0	m_1	m_2	m_2	Cost
*1 B	$\overline{x}\,\overline{y}$	X	X			3
C	$\overline{x}\,\overline{z}$	X		X		3
E	$\overline{x}y$				X	3
G	$\overline{x}y\overline{z}$			X	X	4,5

Row B is the only row covering m_1

$$f_1 = xz + \overline{x}\,\overline{y} + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_2	m_2	Cost
*1 B	$\overline{x}\,\overline{y}$			3
C	$\overline{x}\,\overline{z}$	X		3
E	$\overline{x}y$		X	3
G	$\overline{x}y\overline{z}$	X	X	4,5

Delete columns m_0, m_1

$$f_1 = xz + \overline{x}\,\overline{y} + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_2	m_2	Cost
C	$\overline{x}\,\overline{z}$	X		3
E	$\overline{x}y$		X	3
G	$\overline{x}y\overline{z}$	X	X	4,5

Delete columns m_0, m_1

$$f_1 = xz + \overline{x}\,\overline{y} + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_2	m_2	Cost
C	$\overline{x}\,\overline{z}$	X		3
E	$\overline{x}y$		X	3
G	$\overline{x}y\overline{z}$	X	X	4,5

Cannot delete dominated rows since their cost is lower.
Table is cyclic

$$f_1 = xz + \overline{x}\,\overline{y} + \cdots$$
$$f_2 = z + \cdots$$

Table Reduction

		m_2	m_2	Cost
C	$\overline{x}\,\overline{z}$	X		3
E	$\overline{x}y$		X	3
G	$\overline{x}y\overline{z}$	X	X	4,5

Cannot delete dominated rows since their cost is lower.
Table is cyclic

$$f_1 = xz + \overline{x}\,\overline{y} + \overline{x}y\overline{z}$$
$$f_2 = z + \overline{x}y\overline{z}$$