Class Exercise

9/14/15

Axioms and Theorems:

P1. Closure

For all $x, y \in B, x + y \in B, x \cdot y \in B$

P2. Identity

There exist identity elements in B, denoted 0,1 relative to (+) and (·), respectively.

For all $x \in B, 0 + x = x + 0 = x, 1 \cdot x = x \cdot 1 = x$.

P3. Commutativity

The operations (+), (·) are commutative

For all $x, y \in B\ x + y = y + x,\ \ x \cdot y = y \cdot x$

P4. Distributivity

Each operation (+), (·) is distributive over the other.

For all $x, y, z \in B$:

$$x + (y \cdot z) = (x + y) \cdot (x + z)\ [x\ OR\ (y\ AND\ z)]$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)\ [x\ AND\ (y\ OR\ z)]$$

P5. Complement

For every element $x \in B$ there exists an element $\overline{x} \in B$called the complement of x such that:

$$x + \overline{x} = 1$$

$$x \cdot \overline{x} = 0$$

P6. Non-triviality

There exist at least two elements $x, y \in B$ such that $x \neq y$.

Theorem 3.2: Null elements:

For each element x in a Boolean algebra:

$$x + 1 = 1$$

$$x \cdot 0 = 0.$$

Theorem 3.4: The Idempotent law

For each element x in a Boolean algebra:

$$x + x = x$$

$$x \cdot x = x$$

Theorem 3.5: The Involution Law

For every x in a Boolean algebra, $\overline{(\overline{x})} = x$.

Theorem 3.6: The absorption law

For each pair of elements x, y in a Boolean algebra:

$$x + x \cdot y = x$$

$$x \cdot (x + y) = x$$

Theorem 3.7: DeMorgan's Law

For each pair of elements x, y in a Boolean algebra:

$$\overline{(x + y)} = \overline{x} \cdot \overline{y}$$

$$\overline{(x \cdot y)} = \overline{x} + \overline{y}$$

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1. Prove $x + \overline{x}y = x + y$

$x + \overline{x}y$
$= (x + \overline{x})(x + y)$ (by P4)
$= 1 \cdot (x + y)$ (by P5)
$= x + y$ (by P2)

2. Prove $\overline{x}\,\overline{y} + xy + \overline{x}y = \overline{x} + y$

$\overline{x}\,\overline{y} + xy + \overline{x}y$
$= \overline{x}\,\overline{y} + y(x + \overline{x})$ (by P4)
$= \overline{x}\,\overline{y} + y \cdot 1$ (by P5)
$= \overline{x}\,\overline{y} + y$ (by P2)
$= y + \overline{y}\,\overline{x}$ (by P3)
$= y + \overline{x}$ (by Problem 1 above)
$= \overline{x} + y$ (by P3)

3. Prove that no Boolean algebra has exactly three distinct elements.

Assume towards contradiction there are three elements, $0, 1, \alpha$. (Note that 0,1 must be elements of the Boolean algebra due to P2).
By P5, α must have a complement, $\overline{\alpha}$ such that: $\alpha \cdot \overline{\alpha} = 0$ and $\alpha + \overline{\alpha} = 1$.
There are three cases:
Case 1: $\overline{\alpha} = 0$. In this case, we have $\alpha + 0 = \alpha$ (by P2) and $\alpha + 0 = \alpha + \overline{\alpha} = 1$ (by above). But now combining the above, we get that $\alpha = 1$, so it is not a distinct element.
Case 2: $\overline{\alpha} = 1$. In this case, we have $\alpha \cdot 1 = \alpha$ (by P2) and $\alpha \cdot 1 = \alpha \cdot \overline{\alpha} = 0$ (by above). But now combining the above, we get that $\alpha = 0$, so it is not a distinct element.
Case 3: $\overline{\alpha} = \alpha$. In this case, we have $\alpha + \alpha = \alpha$ (by Th 3.4) and $\alpha + \alpha = \alpha + \overline{\alpha} = 1$. Combining the above, we get that $\alpha = 1$. We also have $\alpha \cdot \alpha = \alpha$ (by Th 3.4) and $\alpha \cdot \alpha = \alpha \cdot \overline{\alpha} = 0$. Combining the above, we get that $\alpha = 0$.
Contradiction.