Convergence of Dual Algorithm with Arbitrary Communication Delays

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Abstract

We study the issue of convergence of user rates and resource prices under a family of rate control schemes called dual algorithms with arbitrary communication delays. We first consider a case where a single resource is shared by many users. Then, we study a general network shared by heterogeneous users and derive sufficient conditions for convergence. We show that in the case of a single user utilizing a single resource, our condition is also necessary. Using our results we derive a sufficient condition for convergence with a family of popular utility and resource price functions. We present numerical examples to validate our analysis.

1 Introduction

A communication network, e.g., the Internet or a telephone network, comprises networking equipments that enable exchange of information between multiple parties or end user equipments (e.g., personal computers or laptops). Networking equipments include switches, routers and links (e.g., fiber optics or twisted copper wires) that connect switches and routers and have finite bandwidth or capacity, i.e., the number of bits that can be transmitted by a link per unit time. These networking equipments, which we call network elements in this paper, communicate with each other using a set of rules called *network protocols*.

Since the links in the Internet have finite capacity, excessive demands brought on by the users can cause severe congestion or even a collapse of the Internet (e.g., the congestion collapse of 1986). Hence, in order to prevent any unexpected collapse of the Internet from severe congestion, it is imperative to control the congestion level inside the network by regulating the rates at which packets are injected into the network by the users, called the packet transmission rates or simply rates of the users. With the increasing size and complexity of the Internet, the problem of computing a fair share of network bandwidth for every user and

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allocating their rates is becoming a challenging task. To this end, in his seminal work [13] Kelly suggested that the problem of allocating fair shares of available network bandwidth to elastic traffic users[1], which we call a *rate allocation* problem, can be posed as an optimization problem.

Under the proposed optimization framework each user receives a utility as a function of its rate, i.e., its share of bandwidth. The utility of a user can represent either the true utility or benefit the user receives or a utility function enforced by the end user protocol that adjusts the packet transmission rate on behalf of the user. An example of such a protocol is Transmission Control Protocol (TCP), which is the most popular congestion or rate control protocol in the Internet today. In the latter case the selection of the utility function determines the behavior of the end user protocol and the desired rate allocation [14, 17, 19]. The objective of the optimization problem is to maximize the aggregate utility, i.e., the sum of the utilities of the users, subject to the link capacity constraints. Using the proposed framework Kelly and his colleagues proposed two classes of *distributed* rate control algorithms - primal and dual algorithms [14] (described in subsections 2.1 and 2.2) - and established their convergence to the desired rate allocation in the absence of delays.

In reality signals take time to travel from one end of a link connecting two network elements to the other end. This introduces a communication delay, which equals the length of the link divided by the speed at which the signal travels the medium, when a signal is transmitted over a link. Modeling communication delays over links between network elements is important when the delays are non-negligible (e.g., intercontinental links) or widely varying with uncertainty, i.e., the delays are not known in advance. An example of such an environment is multi-hop wireless networks, which are wireless networks formed and maintained by (mobile) users without any infrastructure including physical links [26].

Recently, the convergence of user rates to the desired allocation under the primal algorithm in the presence of communication delays has been studied extensively [4, 12, 23, 29, 32]. The authors of [4, 12, 23] provided sufficient conditions on the gain parameters of the users[2] and communication delays for convergence, whereas Ranjan et al. [29] studied the case with arbitrary communication delays and provided sufficient conditions on user utility functions and resource price functions for convergence. Throughout this paper a resource refers to a link that connects two network elements. In addition, the authors of [29, 32] provided sufficient conditions for convergence with popular utility and resource price functions [2].

The convergence of user rates and resource prices under the dual algorithm in the presence of communication delays, however, has not been studied as much. Maulloo [22] studied the local convergence of the dual algorithm using a linearized model and provided sufficient conditions on the delay and resource gain parameters for local convergence. Low et al. introduced a family of dual algorithms, which are variants of those proposed by Kelly et al. [14], and studied the convergence in the presence of communication delays [1, 25].

[1] A user or a connection created by the user for information exchange is said to be *elastic* if its packet transmission rate can be adjusted based on feedback from the network (e.g., packet losses).

[2] The gain parameter of a user or a resource determines how fast the user changes its rate or the resource updates its price in response to a change in network congestion level and is explained in subsections 2.1 and 2.2.

In this paper we study the convergence property of the dual algorithm proposed by Kelly et al. [14] and derive sufficient conditions for convergence with *arbitrary* communication delays. We use the same technique we employed in [29] for the primal algorithm and demonstrate that the same framework can be used to investigate both the primal algorithm and the dual algorithm. We first consider a simpler scenario where a single resource is shared by a large number of users with heterogeneous round-trip delays. We model the dispersion or spread of heterogeneous round-trip delays of many users utilizing the resource using a family of probability distributions known as gamma kernels. This family of distributions has been used to model delay dispersion in other disciplines (e.g., [3, 31]). Then, using the MacDonald's linear chain technique [20] we write the system dynamics as higher order differential equations [7]. We derive a sufficient condition for convergence of user rates and resource prices. We also study the case where the users have the same round-trip delay (i.e., a discrete delay). We show that when the derived sufficient condition is violated, the system becomes unstable for sufficiently large delays and exhibits oscillatory behavior. Using a linear analysis we provide an upper bound on the delay for local convergence.

We extend the above results to general network cases where a set of resources is shared by heterogeneous users, and derive sufficient conditions for convergence in the presence of arbitrary communication delays. These sufficient conditions are derived based on a simple discrete time map that emerges from the intrinsic market structure that underlies the rate control system and captures the interaction between the demands of the users and supplies of the network resources. We note that a similar approach has been used in [10, 11, 21] in the past to study the behavior of delay-differential systems. We apply our results to derive (necessary and) sufficient conditions with the same utility and resource price functions studied in [29, 32]. We show that the derived conditions for the dual algorithm are less restrictive than those for the primal algorithm in [29, 32]. In other words, for some choices of users' utility and resource price functions the dual algorithm converges irrespective of the communication delays, while the convergence of the primal algorithm with the same utility and resource price functions can be ensured only for small delays.

The main contributions of this paper can be briefly summarized as follows:

- We provide sufficient conditions for convergence of the dual algorithm, which are robust against the variations in the communication delays and resource gain parameters. This result can be used to provide a guideline and to simplify the design of the rate control system for a communication network that constantly evolves and changes (e.g., the Internet).

- We demonstrate that when the dispersion of round-trip delays can be modeled by a gamma kernel, the effects of the heterogeneous delays can be studied using the model of Hale and Ivanov [7] and are similar to those of introducing low pass filters in the feedback control loop [18].

This paper is organized as follows: Section 2 describes the optimization framework for rate allocation and the primal and dual algorithms proposed by Kelly et al. [13, 14]. We study the simpler case of a single resource in Section 3. This is followed by a study of general network cases in Section 4. We apply our results to example utility and resource price functions and derive (necessary and) sufficient conditions for convergence with arbitrary communication delays in Section 5. Simulation results are provided in Section 6. We conclude in Section 7.

2 Background

In this section we briefly describe the rate allocation problem in the proposed optimization framework. Consider a network with a set $\mathcal{L}$ of resources and a set $\mathcal{I}$ of users. Let C_l denote the finite capacity of resource $l \in \mathcal{L}$. Each user $i \in \mathcal{I}$ has a fixed route r_i, which is a set of resources traversed by user i's packets. We define a zero-one matrix A, where $A_{i,l} = 1$ if $l \in r_i$ and $A_{i,l} = 0$ otherwise. When its rate is x_i, user i receives utility $U_i(x_i)$. We take the view that the utility functions of the users are used to select the desired rate allocation among the users. The utility $U_i(x_i)$ is an increasing, strictly concave and continuously differentiable function of x_i over the range $x_i \geq 0$. Under this setting, the rate allocation problem of interest can be formulated as the following optimization problem [13]:

SYSTEM(U,A,C):

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in \mathcal{I}} U_i(x_i) \\ \text{subject to} \quad & A^T x \leq C \ , \ x \geq 0 \end{aligned} \tag{1}$$

where $C = (C_l, l \in \mathcal{L})$.[3] The first constraint is the capacity constraint which states that the sum of the rates of all users utilizing resource l should not exceed its capacity C_l.

With the goal of solving this optimization problem in a *distributed* manner, Kelly et al. proposed two classes of rate-based algorithms [14]: Suppose that every user adopts rate-based congestion control in that it adjusts its rate based on the feedback from the network in the form of resource prices. Let $w_i(t)$ and $x_i(t)$ denote the amount user i is willing to pay, which we call its willingness to pay, per unit time and its rate at time t, respectively.[4] At time t each resource $l \in \mathcal{L}$ charges a price per unit flow of $\mu_l(t)$.

2.1 Primal algorithm

In a primal algorithm the end users adjust their rates based on the (shadow) prices per unit time of the resources given by

$$\mu_l(t) = p_l\Big(\sum_{i: l \in r_i} x_i(t) \Big) \ , \ l \in \mathcal{L}, \tag{2}$$

[3]All vectors are assumed to be column vectors.

[4]Throughout the rest of the paper we refer to the willingness to pay per unit time as simply willingness to pay.

where $p_l(\cdot)$ is an increasing function of the aggregate rate of the users going through it. Based on the resource prices, each user i adjusts its rate according to the following differential equation.

$$\frac{d}{dt}x_i(t) = \kappa_i\Big(w_i(t) - x_i(t)\sum_{l\in r_i}\mu_l(t)\Big), \qquad i \in \mathcal{I}, \tag{3}$$

where $w_i(t) = x_i(t) \cdot U_i'(x_i(t))$ and the user gain parameter $\kappa_i > 0$. The basic idea in (3) is to provide a market based rate control mechanism; each user i constantly attempts to reach a market equilibrium where its willingness to pay $w_i(t)$ equals its total price per unit time charged by the resources given by $x_i(t)\sum_{l\in r_i}\mu_l(t)$. Note that the prices charged by the resources in (2) depend on the rates of the users, which can be viewed as users' current demands.

Under (3) one can see that both users' utility functions and resource price functions can be utilized to decide a desired allocation of network bandwidth to the end users. Therefore, the design of rate control algorithms is equivalent to selecting users' utility functions and the price functions of the resources that appear in (2) and (3).

Kelly et al. [14] have shown that, under some conditions on $p_l(\cdot), l \in \mathcal{L}$, the user rates $x(t) = (x_i(t), i \in \mathcal{I})$ converge to a rate vector that maximizes the following expression:

$$\mathcal{U}(x) = \sum_i U_i(x_i) - \sum_l \int_0^{\sum_{i:l\in r_i} x_i} p_l(y)dy \tag{4}$$

Note that the first term in (4) is the aggregate utility of the users in our *SYSTEM(U, A, C)* problem in (1) which we want to maximize. Thus, the primal algorithm proposed by Kelly et al. in (3) solves a variation of the *SYSTEM(U,A,C)* problem in that it maximizes (4) instead of the aggregate utility in the original *SYSTEM(U, A, C)* problem.

2.2 Dual algorithm

In a dual algorithm each resource $l \in \mathcal{L}$ updates its price, $\mu_l(t)$, based on the difference between the observed aggregate rate of the users and its *expected* rate $q_l(\mu_l(t))$ according to

$$\frac{d}{dt}\mu_l(t) = \kappa_l\Big(\sum_{j:l\in r_j} x_j(t) - q_l(\mu_l(t))\Big), \tag{5}$$

where the resource gain parameter $\kappa_l > 0$. The user rates are set to

$$x_j(t) = \frac{w_j(t)}{\sum_{l\in r_j}\mu_l(t)} =: D_j\Big(\sum_{l\in r_j}\mu_l(t)\Big), \tag{6}$$

where $D_j(\lambda)$ is the solution to $\lambda = U_j'(x)$ with $D_j(\lambda) = 0$ if $\lambda \geq U_j'(0)$ and $D_j(\lambda) = \infty$ if $\lambda \leq U_j'(\infty)$.[5] In other words, $D_j(\lambda)$ denotes the demand of user j as a function of the price per unit flow λ, which is the solution to the user optimization problem $\max_{x\geq 0} U_j(x) - x\cdot\lambda$ (called USER(U_j; λ) problem [13]).

[5] One should interpret $U_j'(\infty)$ to be $\lim_{x\to\infty} U_j'(x)$.

We call D_j the demand function of user j throughout the paper. It is easy to see that if $(U'_j)^{-1}$ exists, then $D_j(\lambda) = (U'_j)^{-1}(\lambda)$.

Here, the function q_l can be viewed as the inverse function of the resource price function p_l in the primal algorithm. Hence, $q_l(\mu_l(t))$ gives the expected rate of resource l given its current price $\mu_l(t)$. From (5) - (6) one can see that each resource adjusts its price according to the difference between the user demand (given by aggregate rate $\sum_{j:l\in r_j} x_j(t)$) and its desired supply at the current price (given by $q_l(\mu_l(t))$) [14, 30].

Kelly et al. [14] have proved that under mild technical conditions on the functions $q_l, l \in \mathcal{L}$, the expression

$$\mathcal{V}(\mu) \;=\; \sum_{i\in\mathcal{I}} \int_0^{\sum_{l\in r_i}\mu_l} D_i(\xi)\, d\xi - \sum_{l\in\mathcal{L}} \int_0^{\mu_l} q_l(\eta)\, d\eta \tag{7}$$

provides a Lyapunov function for the system of differential equations (5) - (6). We call a resource price vector μ that maximizes (7) a solution to (7) in the rest of the paper. Similarly, we call a rate vector x that maximizes (4) a solution to (4).

It is a simple exercise to show that if $q_l(\cdot) = p_l^{-1}(\cdot)$, at the solution to (7) denoted by $\overline{\mu}^\star = (\mu_l^\star, l \in \mathcal{L})$, (i) the solutions to USER$(U_i;\ \sum_{l\in r_i} \mu_l^\star)$ problems are the solution to (4) $\overline{x}^\star = (x_i^\star, i \in \mathcal{L})$, and (ii) $p_l(\sum_{i:l\in r_i} x_i^\star) = \mu_l^\star$ for all $l \in \mathcal{L}$. In other words, the user rates and resource prices at the equilibrium are the same under both the dual algorithm and the primal algorithm.

3 Single-Resource Case

In this section we first study a simpler case where a single resource is shared by users with the same utility function. This is similar to the model used in [9] for studying the interaction of TCP connections with a Random Early Detection (RED) gateway.[6] However, unlike in [9] we do not assume that the round-trip delays of the users are the same. General network cases will be studied in the following section.

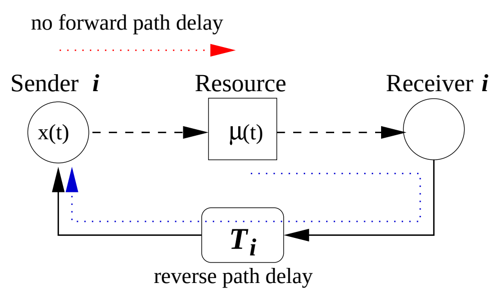


Figure 1: Network model for a single resource case.

[6] A RED gateway is a queue management scheme that attempts to regulate the rates of the users by either dropping or marking packets with some probability to signal to the users impending congestion. The drop or mark probability is a function of the average number of packets queued at the gateway over a period.

We assume that there is no forward delay from senders to the resource, and all of user i's round-trip delay, denoted by T_i, lies in the reverse path from the resource back to the sender. This is shown in Fig. 1. Under this assumption the resource price update rule in (5) is given by

$$\frac{d}{dt}\mu(t) \quad = \quad \kappa\Big(\sum_{i\in\mathcal{I}} \frac{w_i(t)}{\mu(t-T_i)} - q(\mu(t))\Big)\ . \tag{8}$$

We first focus on the case where users' willingness to pay is fixed, i.e., $w_i(t) = w, w > 0$, for all $i \in \mathcal{I}$. The case with user adaptation is discussed in subsection 3.3.

Here we are interested in the case where the resource is shared by a large number of users, e.g., an inter-continental link. In order to facilitate the analysis we assume that we can model the dispersion of heterogeneous round-trip delays of the users using some distribution function $\bar{K}$ as follows: Suppose $T \geq 0$ is the minimum round-trip delay of the users. For every $u \in [0,\infty)$, let $\bar{K}(u)$ be the *fraction* of users whose round-trip delays are less than or equal to $u+T$. We assume $\bar{K}$ is differentiable and yields a density function K, i.e., $K(u) = \bar{K}^{(1)}(u)$. This is reasonable when the number of users is large. Under this assumption, we can approximate the *average* rate of the users seen at the resource at time t using

$$\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}}\frac{w}{\mu(t-T_i)} \approx \int_0^\infty \frac{w}{\mu(t-T-s)}K(s)ds\ , \tag{9}$$

where $|\mathcal{I}|$ is the cardinality of $\mathcal{I}$.

In the rest of this section we normalize both the aggregate rate of the users at the resource and the expected rate of the resource in (8) by the number of users $|\mathcal{I}|$, and replace these terms with (9) and the expected rate per user of the resource, respectively.

$$\begin{aligned} \frac{d}{dt}\mu(t) \quad &= \quad \overline{\kappa}\left(\int_0^\infty \frac{w}{\mu(t-T-s)}\ K(s)\ ds - q_N(\mu(t))\right)\ , \\ &= \quad \overline{\kappa}\left(\int_0^\infty f(\mu(t-T-s))\ K(s)\ ds - q_N(\mu(t))\right)\ , \end{aligned} \tag{10}$$

where $\overline{\kappa} = \kappa \cdot |\mathcal{I}|$, $q_N(\mu) = q(\mu)/|\mathcal{I}|$, and $f(\mu) = \frac{w}{\mu}$. Hence, the resource adjusts its price based on the average rate of the users and its expected (average) user rate based on the current price $\mu(t)$.

In this paper we consider the case where the delay density function K can be modeled by a family of generic delay kernels also known as gamma kernels, which have the following form:

$$K(u) \quad = \quad \begin{cases} \frac{\alpha^{r+1}u^r}{r!}\exp(-\alpha u) & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}\ , \tag{11}$$

where $\alpha > 0$ is a constant, and $r \in \{0, 1, 2, \ldots\}$. The kernel K with different parameters is plotted in Fig. 2.

The parameter r is called the order of gamma kernel K [20], and the mean of K for fixed (α, r) is given by

$$\mathbf{E}\,[K] \quad = \quad \int_0^\infty u\frac{\alpha^{r+1}u^r}{r!}\ \exp(-\alpha u)\ du = \frac{r+1}{\alpha}\ .$$

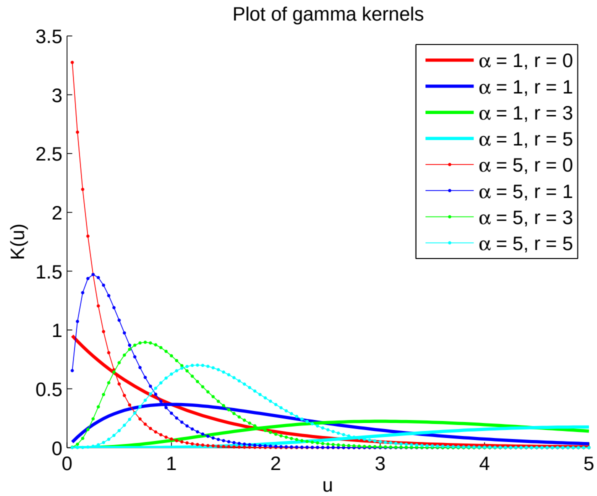


Figure 2: Gamma kernels for different parameters.

The kernel K with $r = 0$ and $r = 1$ is called the weak and strong kernel, respectively, and is frequently used to model distributed delay in different disciplines including population biology [3, 31]. These gamma kernels can be used to model a whole class of delay distributions (e.g., Fig. 2), including an exponential distribution (i.e., the weak kernel). The case of discrete delay can be realized by letting r and α go to infinity simultaneously while keeping the mean delay $\frac{r+1}{\alpha}$ fixed.

In the remainder of this section we study the asymptotic behavior of the system in (10) under a set of reasonable assumptions on function q. In particular, we adopt the gamma kernels in (11) and apply the *MacDonald's linear chain technique* to derive sufficient conditions for convergence of the dual algorithm [20].

Define $\mathbb{R}_+ := (0, \infty)$ and $\overline{\mathbb{R}}_+ := [0, \infty)$. The resource price function p is a strictly increasing function that maps $\overline{\mathbb{R}}_+$ to $\overline{\mathbb{R}}_+$. We first introduce the following assumption.

Assumption 1 *(i) The function $q : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is strictly increasing with $q(0) = 0$. (ii) The function q is Lipschitz continuous on any bounded interval $[\mu_{min}, \mu_{max}] \subset \mathbb{R}_+$. (iii) There exists a unique point $\mu^\star \in \mathbb{R}_+$ such that $f(\mu^\star) = q(\mu^\star)/|\mathcal{I}| = q_N(\mu^\star)$.*

It is clear from (10) that $\mu^\star$ is the unique equilibrium of the dual algorithm in (5), at which the total demand of the users equals the expected rate of the resource given by $q(\mu^\star)$. Note that under Assumption 1, if the initial function is non-negative, i.e., $\mu(s) \geq 0$ for all $s \in (-\infty, 0)$, the resource price $\mu(t)$ remains non-negative because when $\mu(t) = 0$, the right-hand side of (10) is non-negative if $\mu(t') \geq 0$ for all $t' < t$.

The existence of a unique solution of (10) is guaranteed by Theorem 2.3 in [8, p. 44] under Assumption 1 if there exist bounds μ_{min} and μ_{max}, where $0 < \mu_{min} < \mu_{max} < \infty$, such that $\mu(t)$ remains in $[\mu_{min}, \mu_{max}]$ for all $t \geq 0$, starting with an appropriate initial function that lies in $[\mu_{min}, \mu_{max}]$. In the following section we assume the existence of a unique positive solution of (10) and the bounds μ_{min} and μ_{max} such that

$\mu_{min} \le \mu(t) \le \mu_{max}$ for all $t \ge 0$. We will provide an assumption under which this is true in subsection 3.2.

3.1 MacDonald's linear chain technique

Let $\mu(t)$ be a positive solution of (10) with some positive initial function $\mu(s)$ for all $s < 0$. We define

$$\begin{aligned} \omega_i(t) &= \int_0^\infty \frac{w}{\mu(t-T-s)}\, G^i(s)\, ds \\ &= \int_{-\infty}^t f(\mu(\theta - T))\, G^i(t-\theta)\, d\theta, \qquad i = 0, 1, \ldots, r\ , \end{aligned} \tag{12}$$

where $\theta = t - s$, $d\theta = -ds$ and $G^i(u) = \frac{\alpha^{i+1} u^i}{i!} \exp(-\alpha u)$, $u \ge 0$. Note that for any $i \in \{1, 2, \ldots\}$,

$$\frac{d}{du} G^i(u) = -\alpha G^i(u) + \alpha G^{i-1}(u)\ , \text{ and } \frac{d}{du} G^0(u) = -\alpha G^0(u)\ . \tag{13}$$

Suppose that the delay kernel K is given by G^r for some $\alpha > 0$ and $r \in \{0, 1, \ldots\}$. From (13) we see that $(\mu(t), \omega_r(t), \omega_{r-1}(t), \ldots, \omega_0(t))$ satisfies

$$\frac{d}{dt}\mu(t) = \overline{\kappa}\left(\omega_r(t) - q_N(\mu(t))\right) \tag{14}$$

$$\frac{d}{dt}\omega_i(t) = -\alpha\left(\omega_i(t) - \omega_{i-1}(t)\right)\ ,\ i = 1, \ldots, r \tag{15}$$

$$\frac{d}{dt}\omega_0(t) = -\alpha\left(\omega_0(t) - f(\mu(t-T))\right) \tag{16}$$

Define $\eta(t) := q_N(\mu(t))$. We have

$$\frac{d}{dt}\eta(t) = q_N'(q_N^{-1}(\eta(t)))\frac{d}{dt}\mu(t) \text{ and, hence, } \frac{d}{dt}\mu(t) = \frac{1}{q_N'(q_N^{-1}(\eta(t)))}\frac{d}{dt}\eta(t)\ ,$$

where the inverse q_N^{-1} exists from Assumption 1. Note that, from Assumption 1, if $\mu(t)$ is positive for all $t \ge 0$, so is $\eta(t)$.

We can rewrite (14) in terms of $\eta(t)$ and $\omega_r(t)$ defined in (12).

$$\frac{1}{\overline{\kappa} \cdot q_N'(q_N^{-1}(\eta(t)))}\frac{d}{dt}\eta(t) = \omega_r(t) - \eta(t) \tag{17}$$

Using the definition of $\eta(t)$, we can rewrite (16) as

$$\begin{aligned} \frac{d}{dt}\omega_0(t) &= -\alpha\, \omega_0(t) + \alpha\, f(q_N^{-1}(\eta(t-T))) \\ &= -\alpha\, \omega_0(t) + \alpha\, \check{F}(\eta(t-T))\ . \end{aligned}$$

where the map $\check{F} : \overline{\mathbb{R}}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$\check{F}(\eta) := f(q_N^{-1}(\eta)) = \frac{w}{q_N^{-1}(\eta)}\ , \quad \forall\, \eta \in \overline{\mathbb{R}}_+\ . \tag{18}$$

Therefore, we have the following set of differential equations for describing the dynamics of the rate control system.

$$\begin{aligned}
&\frac{1}{\overline{\kappa} \cdot q_N'(q_N^{-1}(\eta(t)))} \frac{d}{dt}\eta(t) = -\eta(t) + \omega_r(t) \\
&\frac{d}{dt}\omega_i(t) = -\alpha\, \omega_i(t) + \alpha\, \omega_{i-1}(t)\ , \qquad i = 1, \ldots, r \\
&\frac{d}{dt}\omega_0(t) = -\alpha\, \omega_0(t) + \alpha\, \check{F}(\eta(t-T))
\end{aligned} \tag{19}$$

We note that the delay differential system in (19) is similar to that in [18] for a variant of the primal algorithm where the price charged by a resource is a function of the averaged or low pass filtered version of the aggregate rate of the users through the resource, as opposed to the users' instantaneous rates in the original primal algorithm. Hence, this suggests that the dynamical effect of heterogeneous delays in the dual algorithm, summarized by the averaging in (9) is similar to that of low pass filtering (or averaging) of the rate seen by the resource in the primal algorithm [18].

In the rest of this section we will show that, similarly as in the primal algorithm case [29], the convergence of dual algorithm in (10) (or in (19)) can be studied using the map $\check{F}$ defined in (18).

3.2 Convergence

We denote by $C([-T,0],A)$ the Banach space of the continuous functions from the interval $[-T,\ 0]$ to some interval $A \subset \mathbb{R}_+$ with topology of uniform convergence [8]. Suppose that there exists an interval $J := [a,\ b] \subset \mathbb{R}_+$, which is invariant under the map $\check{F}$, i.e., $\check{F}(J) \subset J$. Let $Y_J := C([-T,0],J)$, and the initial function of $\eta(s), s \in [-T,0]$, is given by ϕ.

Theorem 1 *If the initial function $\phi \in Y_J$ and $\omega_0(0), \ldots, \omega_r(0) \in J$, then for all $t \geq 0$ we have $(\eta(t;\phi)$, $\omega_0(t)$, ..., $\omega_r(t)) \in J^{r+2}$.*

Proof: A proof is provided in Appendix A. ■

Theorem 1 implies that, under the assumption stated in the theorem, since $\eta(t;\phi)$ remains in J, from the definition of $\eta(t)$ and Assumption 1 the resource price $\mu(t)$ lies in a compact interval not including zero and, hence, stays away from zero for all $t \geq 0$. Therefore, the existence of a unique solution is guaranteed by Theorem 2.3 in [8, p. 44] under the assumption as mentioned earlier.

For any interval A let $int(A)$ denote its interior. We establish the convergence of (19) under the following assumption.

Assumption 2 *There is a sequence of closed intervals $J_k \subset \mathbb{R}_+, k = 0,1,\ldots$ such that (i) $\check{F}(J_k) \subset int(J_{k+1}) \subset J_{k+1} \subset int(J_k)$ for all $k = 0,1,\ldots$, and (ii) $\cap_{k\geq 0} J_k = \{q_N(\mu^\star)\}$, where $\mu^\star$ is the unique point that satisfies $f(\mu^\star) = q_N(\mu^\star)$ in Assumption 1.*

An example of utility and resource price functions that satisfy this assumption will be given in Section 5.

Theorem 2 *Suppose that Assumption 2 holds. If* $\phi \in Y_{J_0}$ *and* $\omega_i(0) \in J_0$ *for all* $i = 0, 1, \ldots, r$*, then* $(\eta(t;\phi), \omega_0(t), \ldots, \omega_r(t)) \to (q_N(\mu^\star), \ldots, q_N(\mu^\star))$ *as* $t \to \infty$*.*

Proof: A proof is given in Appendix B. ■

3.3 User adaptation

In the previous subsection we have assumed that the willingness to pay of the users is fixed at w. This describes a case where the users' utility function is given by $w \cdot \log(x)$ [12, 14]. Suppose that the users' utility function is not of the form $w \cdot \log(x)$. If the user can accurately track the price per unit flow $\mu(t)$ and solve the USER(U; $\mu(t)$) problem, it should select a rate $x^\star(\mu(t))$ that satisfies $U'(x) = \mu(t)$, and set its willingness to pay to $w(t) = \mu(t) \cdot x^\star(\mu(t))$. This rate $x^\star(\mu(t))$ is given by the demand function D of the user defined in subsection 2.2.

We assume that the demand function D is (i) strictly decreasing in μ on the interval $[U'(\infty), U'(0))$, (ii) differentiable on $(U'(\infty), U'(0))$, and (iii) Lipschitz continuous on every bounded interval $[U'(\mu_{max}), U'(\mu_{min})]$, where $[\mu_{min}, \mu_{max}] \subset \mathbb{R}_+$. An example of utility functions that satisfy these assumptions is provided in Section 5. Under these assumptions one can show that if there exists a unique $\mu^\star$ such that $D(\mu^\star) = q_N(\mu^\star)$ and the map $\check{F}(\eta) := D(q_N^{-1}(\eta))$ satisfies Assumption 2, then the theorems in subsection 3.2 hold with the same proofs. Note that the function $f(\mu) = w/\mu$ plays the role of the demand function D when the willingness to pay w is constant.

3.4 Local stability with a homogeneous delay

In this subsection we consider the case where the resource is utilized by a single user with a fixed delay T. This user can be viewed as the aggregate of many users with the same round-trip delay T [9]. Recall that the case of discrete delay T can be modeled by gamma kernels by letting r and α go to ∞ simultaneously with a fixed mean delay $\frac{r+1}{\alpha} = T$. Using a linear analysis, we study the local stability of the system around the equilibrium $\mu^\star$.

When a single user utilizes the resource, the resource price is updated according to

$$\frac{d}{dt}\mu(t) = \kappa\left(x(t) - q(\mu(t))\right) = \kappa\left(D(\mu(t-T)) - q(\mu(t))\right) . \tag{20}$$

We rewrite (20) in terms of $\eta(t) = q(\mu(t))$ as

$$\begin{aligned} \frac{d}{dt}\eta(t) &= \kappa \cdot q'(q^{-1}(\eta(t)))\left(D(q^{-1}(\eta(t-T))) - \eta(t)\right) \\ &= \zeta(\eta(t))\left(\check{F}(\eta(t-T)) - \eta(t)\right) \end{aligned} \tag{21}$$

where $\zeta(\eta(t)) := \kappa \cdot q'(q^{-1}(\eta(t)))$ and $\check{F}(\eta) = D(q^{-1}(\eta))$. Note that $\zeta(\eta(t)) > 0$ from the assumptions on function q. Following the similar steps in the proof of Theorem 2 one can show that the resource price generated by (21) converges if Assumption 2 holds and $\phi \in Y_{J_0}$.

Assuming that the map $\check{F}$ is locally smooth around $\eta^\star = q(\mu^\star)$, one can find the conditions for local stability of the fixed point $\eta^\star$ for the delay differential equation in (21). Proposition 4 in [24, p. 17] tells us that the following linearized system

$$\begin{aligned}
\frac{d}{dt}Z(t) &= \zeta(\eta(t))\check{F}'(\eta(t-T))\Big|_{\eta=\eta^\star} Z(t-T) \\
&\quad + \Big(\zeta'(\eta(t))\left[\check{F}(\eta(t-T)) - \eta(t)\right] - \zeta(\eta(t))\Big)\Big|_{\eta=\eta^\star} Z(t) \\
&= \zeta(\eta^\star)\check{F}'(\eta^\star)Z(t-T) - \zeta(\eta^\star)Z(t) \\
&:= -B \cdot Z(t-T) - A \cdot Z(t)\,, \qquad (22)
\end{aligned}$$

where $A = \zeta(\eta^\star)$ and $B = -\zeta(\eta^\star)\check{F}'(\eta^\star)$,[7] is stable *if and only if*

(i) $A + B > 0$ and $A \geq |B|$, or

(ii) $B > |A|$ and $T \leq T^\star := \cos^{-1}(-A/B)/\sqrt{B^2 - A^2}$.

Note that in our problem the above conditions tell us that the linearized system in (22) is stable if and only if (i) the map $\check{F}$ is locally stable, i.e., $|\check{F}'(\eta^\star)| < 1$, or (ii) $\check{F}'(\eta^\star) < -1$ and

$$T \leq \frac{\cos^{-1}\left((\check{F}'(\eta^\star))^{-1}\right)}{\zeta(\eta^\star)\sqrt{\left(\check{F}'(\eta^\star)\right)^2 - 1}}\,.$$

Since local stability is required for global stability, these conditions tell us that the local stability of the map $\check{F}$ is both *necessary* and *sufficient* for convergence of the dual algorithm in (20) with an arbitrary delay in the neighborhood of the equilibrium point $\mu^\star$. We use these conditions to establish a *necessary* and *sufficient* condition for convergence with example utility and resource price functions in Section 5.

4 General Network Cases

In the previous section we studied the case where a single resource is shared by many users. Using the MacDonald's linear chain technique we demonstrated the effects of heterogeneous delays of the users are similar to introducing low pass filters in the feedback loop (eq. (19)). Then, we showed that the asymptotic stability of the discrete time map $\check{F}$ is sufficient for convergence of dual algorithm. In this section we extend these results to the case of a general network shared by multiple heterogeneous users with different delays. We first describe the model used for our analysis and then derive sufficient conditions for convergence of the resource prices and user rates with a general network topology.

[7]This is because $\eta^\star$ is a fixed point of the map $\check{F}$, i.e., $\check{F}(\eta^\star) = \eta^\star$.

4.1 General network model with delays

In this subsection we describe the network model that captures the communication delays between network elements and end users under the assumption that the delays are *constant*. Recall from Section 2 that $\mathcal{I} = \{1, \dots, N\}$ is the set of users sharing a network consisting of a set $\mathcal{L}$ of resources. Define $I_l = \{i \in \mathcal{I} \mid l \in r_i\}$ to be the set of users utilizing resource $l \in \mathcal{L}$. For all $i \in \mathcal{I}$ and $l \in r_i$ let $T^r_{i,l}$ and $T^f_{i,l}$ denote the delay of the feedback signal from resource l to sender i and the delay from sender i to resource l, respectively. This is shown in Fig. 3. If user i packets do not traverse resource l, i.e., $l \notin r_i$, we assume that $T^r_{i,l} = T^f_{i,l} = 0$. Suppose that the resources in $r_i = \{l_{i,1}, \dots, l_{i,R_i}\}$ are arranged in the order user i packets visit, where $R_i = |r_i|$. Define $T_i = T^f_{i,l_{i,k}} + T^r_{i,l_{i,k}}, k = 1, \dots, R_i$, to be the round-trip delay of user i.

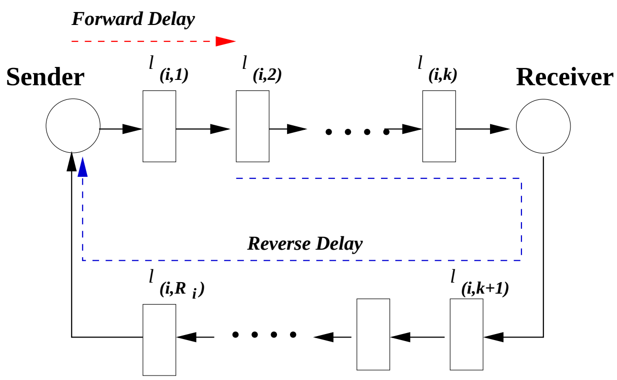


Figure 3: Network model with delays.

Similarly as in the single resource case, we introduce the following assumptions on the demand functions $D_i(\cdot)$ and $q_l(\cdot)$.

Assumption 3 *(i) The demand functions $D_i(\mu)$ are strictly decreasing in price per unit flow μ on the interval $[U'_i(\infty), U'_i(0))$ and differentiable on $(U'_i(\infty), U'_i(0))$. In addition, they are Lipschitz continuous on any bounded interval $A \subset (U'_i(\infty), U'_i(0))$. (ii) The function $q_l : [0, \infty) \to [0, \infty)$ is strictly increasing with $q_l(0) = 0$ for all $l \in \mathcal{L}$. Moreover, the function q_l is Lipschitz continuous on any bounded interval $[\mu_{l,min}, \mu_{l,max}] \subset \mathbb{R}_+$.*

Assumption 3(ii) simply says that the equilibrium price of a resource increases with the aggregate rate traversing the resource, i.e., the total demand from the users. Note that Assumption 3(ii) also ensures that the inverse functions q_l^{-1} exist.

With the communication delays defined above, under Assumption 3, the differential equations in (5) and

(6) become[8]

$$\frac{d}{dt}\mu_l(t) \;=\; \kappa_l \left(\sum_{i\in I_l} D_i\Big(\sum_{j\in r_i} \mu_j(t-(T^r_{i,j}+T^f_{i,l}))\Big) - q_l(\mu_l(t)) \right) \quad \forall\, l \in \mathcal{L}\,. \tag{23}$$

The existence of a unique solution of (23) under Assumption 3 is again guaranteed by Theorem 2.3 in [8, p. 44] if, for all $l \in \mathcal{L}$, the resource price $\mu_l(t)$ remains in some compact set $[\mu_{l,min}, \mu_{l,max}] \subset \mathbb{R}_+$ for all $t \geq 0$. We will provide an assumption (Assumption 4) under which this holds in the following subsection.

Similarly as in the previous section we define $\eta_l(t) := q_l(\mu_l(t))$. Recall that $\eta_l(t)$ denotes the expected rate of resource l at time t as a function of its price $\mu_l(t)$. We rewrite (23) in terms of $\eta_l(t)$ as

$$\frac{d}{dt}\eta_l(t) \;=\; \kappa_l q_l^{'}(q_l^{-1}(\eta_l(t))) \left(\sum_{i\in I_l} D_i\Big(\sum_{j\in r_i} q_j^{-1}(\eta_j(t-T^r_{i,j}-T^f_{i,l}))\Big) - \eta_l(t) \right)\,. \tag{24}$$

This can be put in the following matrix form.

$$\frac{d}{dt}\overline{\eta}(t) \;=\; \overline{\zeta}(t)\,[F(\tilde{\eta}(t)) - \overline{\eta}(t)] \tag{25}$$

where $\overline{\eta}(t) = (\eta_l(t); l \in \mathcal{L})$, $\overline{\zeta}(t) = \text{diag}(\kappa_l \cdot q_l^{'}(q_l^{-1}(\eta_l(t))); l \in \mathcal{L})$, $\tilde{\eta}(t) = (\eta_{(i,l)}(t); l \in \mathcal{L}, i \in I_l)$, and $\eta_{(i,l)}(t) = (\eta_j(t - T^r_{i,j} - T^f_{i,l}); j \in \mathcal{L})$. The l-th element of the multi-dimensional map $F : \overline{\mathbb{R}}_+^{L\cdot\Xi} \to \overrightarrow{\mathbb{R}}_+^L$, where $\Xi := \sum_{l\in\mathcal{L}} |I_l|$ and $\overrightarrow{\mathbb{R}}_+ = \overline{\mathbb{R}}_+ \cup \{\infty\}$, is defined by

$$F_l(\tilde{\eta}(t)) \;=\; \sum_{i\in I_l} D_i\Big(\sum_{j\in r_i} q_j^{-1}(\eta_j(t-T^r_{i,j}-T^f_{i,l}))\Big)\,,\; l \in \mathcal{L}\,. \tag{26}$$

Note that Assumption 3 guarantees that the gain matrix $\overline{\zeta}(t)$ is positive definite.

4.2 Convergence

In this subsection we investigate the convergence of resource prices and aggregate rates at the resources generated by (25). More specifically, we will provide sufficient conditions for their convergence regardless of the delays $T^f_{i,j}$ and $T^r_{i,j}$.

Definition 1 *A set $D \subset \mathbb{R}_+^L$ is said to be invariant under the map F if $F(\tilde{\eta}) \in D$ for all $\tilde{\eta} \in D^{\Xi}$, i.e., $\tilde{\eta} = (\eta_{(i,l)}; l \in \mathcal{L}, i \in I_l)$ and $\eta_{(i,l)} \in D$ for all $l \in \mathcal{L}$ and $i \in I_l$. A vector $\overline{\eta}^\star \in \mathbb{R}_+^L$ is said to be a fixed point of F if $F(\overline{\eta}^\star, \ldots, \overline{\eta}^\star) = \overline{\eta}^\star$.*

The invariance of the map F can be interpreted as follows. Suppose that expected rates $\overline{\eta}(t)$ of the resources based on their current prices at time t as well as time delayed values $\eta_{(i,l)}(t)$, $l \in \mathcal{L}$ and $i \in I_l$, belong to the set D. Then, the invariance of the set D implies that $F(\tilde{\eta}(t))$ lies in the set D. Similarly, a

[8] As explained in the single resource case, under Assumption 3, the resource prices $\mu_l(t)$ remain non-negative if the initial functions are non-negative.

fixed point $\overline{\eta}^\star$ means that if $\overline{\eta}(t) = \overline{\eta}^\star$, $\eta_{(i,l)}(t) = \overline{\eta}^\star$ for all $l \in \mathcal{L}$ and $i \in I_l$, then $F(\tilde{\eta}(t)) = \overline{\eta}^\star$. In other words, $\overline{\eta}(t)$ and hence resource prices $\overline{\mu}(t)$ remain constant. One can verify that if $\overline{\eta}^\star$ is a fixed point of F, then $\overline{q}^{-1}(\overline{\eta}^\star) = (q_1^{-1}(\eta_1^\star), \ldots, q_L^{-1}(\eta_L^\star))$ is a solution to (7) from (25), i.e., , $\overline{\eta}^\star = \overline{q}(\overline{\mu}^\star)$ where $\overline{\mu}^\star$ is a solution to (7).

We now state the assumption under which the convergence of (25) is established.

Assumption 4 *Suppose that $\overline{\eta}^\star \in \mathbb{R}_+^L$ is a fixed point of the multidimensional map F. There is a sequence of compact, convex sets $E_k \subset \mathbb{R}_+^L$, $k \geq 0$, such that $F(E_k^\Xi) \subset int(E_{k+1}) \subset E_{k+1} \subset int(E_k)$ and $\cap_{k\geq 0} E_k = \{\overline{\eta}^\star\}$.*

Define $T_{max} = \max_{l\in\mathcal{L}, i\in I_l}(\max_{j\in r_i}(T_{i,j}^r + T_{i,l}^f))$. We denote by $C([-T_{max}, 0], E)$ the Banach space of continuous functions mapping the interval $[-T_{max}, 0]$ into E with topology of uniform convergence [8]. Let $Y_{E_0} = C([-T_{max}, 0], E_0)$ be a subset of initial functions and $\overline{\eta}(t; \phi)$ a solution of (25) constructed using an initial function $\phi \in Y_{E_0}$.

Theorem 3 *All solutions $\overline{\eta}(t;\phi)$ starting with an initial function $\phi \in Y_{E_0}$ remain in E_0 for all $t \geq 0$, and converge to $\overline{\eta}^\star$ as $t \to \infty$ for all $T_{i,j}^r, T_{i,j}^f \in \mathbb{R}_+$.*

Proof: The basic idea of the proof of the theorem is similar to that of Theorem 4 in [29], and a proof is provided in Appendix F. ■

Theorem 3 tells us that the attracting fixed point of the map F is stable in the set E_0. Since $\overline{\mu}(t) = \overline{q}^{-1}(\overline{\eta}(t))$, this tells us that $\overline{\mu}(t) \to \overline{\mu}^\star$ as $t \to \infty$.

4.3 Comparison with a homogeneous delay system

In this subsection we investigate how the convergence of the resource prices under the general delay differential system in (25) is related to that of a much simpler system where (i) there are no forward delays from the senders to the resources and (ii) all users have the same homogeneous round-trip delay. In other words, $T_{i,l}^f = 0$ and $T_{i,l}^r = T$ for all $i \in \mathcal{I}$ and $l \in r_i$, where T is some positive constant. Under this assumption the delay differential equations in (24) simplify to

$$\frac{d}{dt}\eta_l(t) = \kappa_l q_l^{'}(q_l^{-1}(\eta_l(t))) \left(\sum_{i\in I_l} D_i\Big(\sum_{j\in r_i} q_j^{-1}(\eta_j(t-T))\Big) - \eta_l(t) \right) \; \forall\, l \in \mathcal{L} \, , \tag{27}$$

and the matrix form is given by

$$\frac{d}{dt}\overline{\eta}(t) = \overline{\zeta}(t)\left[\hat{F}(\eta(t-T)) - \overline{\eta}(t)\right]$$

where the map $\hat{F} : \overline{\mathbb{R}}_+^L \to \overrightarrow{\mathbb{R}}_+^L$ is defined by

$$\hat{F}_l(\overline{\eta}) = \sum_{i\in I_l} D_i\Big(\sum_{j\in r_i} q_j^{-1}(\eta_j)\Big) \, , \quad l \in \mathcal{L} \, . \tag{28}$$

Assumption 5 *The multidimensional map $\hat{F}$ has a fixed point $\overline{\eta}^\star \in \mathbb{R}^L_+$. In addition, there is a sequence of compact, convex sets $\check{E}_k \subset \mathbb{R}^L_+$, $k \geq 0$, such that $\hat{F}(\check{E}_k) \subset int(\check{E}_{k+1}) \subset \check{E}_{k+1} \subset int(\check{E}_k)$ and $\cap_{k\geq 0}\check{E}_k = \{\overline{\eta}^\star\}$, i.e., the map $\hat{F}$ is stable with a domain of attraction $\check{E}_0$.*

Theorem 4 *All solutions $\overline{\eta}(t;\phi)$ of (27) starting with initial function $\phi \in Y_{\check{E}_0}$ converge to $\overline{\eta}^\star$ as $t \to \infty$ for all $T > 0$.*

Proof: The proof is provided in Appendix G. ∎

Theorem 4 tells us that if $\check{E}_0$ is a region of attraction of the map $\hat{F}$ in (28), then the resource prices under the delay differential system in (27) with a homogeneous delay converge, provided that the initial function lies in $\check{E}_0$. It is easy to show that the same sequence of closed, convex sets $\check{E}_k, k \geq 0$, in Assumption 5 also satisfies Assumption 4. This follows from the assumed monotonicity properties of the functions q_l and D_i stated in Assumption 3. This in turn implies that the resource prices under the delay differential system in (25) converge if the initial function lies in $\check{E}_0$. Hence, the stability of the map $\hat{F}$ is a *sufficient* condition for convergence of resource prices under (25) with arbitrary communication delays.

5 Example Utility and Resource Price Functions

In this section we adopt a family of well known users' utility functions and resource price functions studied in [29, 32], and derive a condition for convergence with arbitrary gains κ_l and delays, making use of our results in Sections 3 and 4. Users' utility functions are of the following form:

$$U_a(x) = \begin{cases} \frac{1}{a}x^a\ , & -\infty < a < 1,\ a \neq 0 \\ \log(x)\ , & a = 0 \end{cases} \tag{29}$$

In particular, $a = -1$ has been found useful for modeling the utility function of TCP-like algorithms [15]. With the utility functions of the form in (29), user i's price elasticity of demand [9] is given by $-1/(1-a)$. Thus, one can see that users become more elastic or responsive with increasing value of a [29], i.e., the sensitivity of user demand, $1/(1-a)$, increases with a. Since the utility function $U_a(x)$ is strictly concave with $\lim_{x\downarrow 0} U_a'(x) = \infty$ and $\lim_{x\uparrow\infty} U_a'(x) = 0$, the demand function $D_a(\mu)$ is well defined for all $\mu \in \mathbb{R}_+$ and is given by

$$D_a(\mu) = \mu^{-1/(1-a)}\ . \tag{30}$$

The class of resource price functions that we are interested in has the form

$$p(x) = q^{-1}(x) = c \cdot \left(\frac{x}{C}\right)^b\ , \quad x \in \overline{\mathbb{R}}_+\ , \tag{31}$$

[9] Price elasticity of demand measures the sensitivity of a user's demand to price changes and is defined to be $\frac{\mu}{D(\mu)}\frac{dD(\mu)}{d\mu}$, where $D(\mu)$ is the demand of the user at the price μ.

where $b > 0$, c is some positive constant, and C is the capacity of the resource. However, C can be replaced by a virtual capacity, typically smaller than the real capacity. The use of virtual capacity was first proposed in [6] to reduce packet losses due to buffer overflow at highly utilized resources, at the expense of slightly reduced utilization. Kunniyur and Srikant in [16] proposed dynamically adjusting the virtual capacity and consequently the resource price function, based on the current aggregate rate seen at the resource. The value of c does not affect our convergence results and is assumed to be one unless stated otherwise. From (31), the function q is given by

$$q(\mu) = C \cdot \mu^{1/b} \ , \ \mu \in \overline{\mathbb{R}}_+ \ . \tag{32}$$

The parameter b is used to change the shape of the price function. The larger b is, the more convex and responsive the price function is. It is easy to verify that user demand function D_a in (30) and resource expected rate function q in (32) satisfy the assumptions in Sections 3 and 4.

5.1 A single user, single resource case

Suppose that the utility function of the user is given by $U_a(x)$, $a < 1$, and $q(\mu)$ is of the form in (32) for some parameter $b > 0$.

Assumption 6 *Suppose that* $a + b < 1$.

Let σ be a constant that satisfies $\frac{b}{1-a} < \sigma < 1$. Since $a + b < 1$, it is easy to see that one can find such σ. Fix $\overline{\alpha} > 1$ and choose $\overline{\beta} < 1$ that satisfies

$$\overline{\alpha}^{-\frac{(1-a)}{b}} < \overline{\beta} < \overline{\alpha}^{-\frac{b}{1-a}} \ . \tag{33}$$

Again, the existence of such $\overline{\beta}$ is guaranteed from the assumption that $\overline{\alpha} > 1$ and $\frac{b}{1-a} < 1$.

Define a sequence of compact intervals $I_k, k \geq 0$, given by

$$I_k \ = \ \begin{cases} [\overline{\beta}^{\sigma^k} \mu^\star, \ \overline{\alpha}^{\sigma^k} \mu^\star] \ , & \text{if } k \text{ is even} \\ [\overline{\alpha}^{-\sigma^k} \mu^\star, \ \overline{\beta}^{-\sigma^k} \mu^\star] \ , & \text{if } k \text{ is odd} \end{cases} \tag{34}$$

where $\mu^\star$ is the unique solution to (7) and is given by $C^{-b(1-a)/(1+b-a)}$. Note that since $0 < \sigma < 1$, $\sigma^k \to 0$ and the interval I_k decreases to $\{\mu^\star\}$ as $k \to \infty$.

We define a map $\overline{F} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+ \cup \{\infty\}$, where $\overline{F}(\mu) = q^{-1}(D_a(\mu))$.

Lemma 1 *If Assumption 6 holds, then we have*

$$\overline{F}(I_k) \subset int(I_{k+1}) \subset I_{k+1} \subset int(I_k) \ \textit{ for all } k \geq 0.$$

Proof: A proof is provided in Appendix C. ∎

Theorem 5 *Suppose that* $a + b < 1$ *and the initial function* $\phi \in C([-T, 0], I_0 = [\overline{\beta}\mu^\star,\ \overline{\alpha}\mu^\star])$ *with any* $\overline{\alpha} > 1$ *and* $\overline{\beta} < 1$ *satisfying (33). Then, the solution* $\mu(t; \phi)$ *of (20) converges to* $\mu^\star$ *as* $t \to \infty$*.*

Proof: The theorem follows directly from Theorem 2 and Lemma 1. ∎

Note that as $\overline{\alpha} \uparrow \infty$, $\overline{\beta} \downarrow 0$. Since the above is true for any arbitrary $\overline{\alpha} > 1$ and $\overline{\beta}$ that satisfies (33), the resource price $\mu(t)$ converges to $\mu^\star$ starting from any arbitrary positive value because I_0 can be made large enough to contain the initial function.

We now show that if $a + b > 1$, then the system is unstable for sufficiently large T. Note that the map $\check{F}(\eta)$ is given by

$$\check{F}(\eta) = D_a(q^{-1}(\eta)) = C^{b/(1-a)} \eta^{-b/(1-a)} \ .$$

Hence,

$$\check{F}'(\eta)\Big|_{\eta=\eta^\star} = \frac{-b \cdot C^{b/(1-a)}}{1-a} (\eta^\star)^{-(1+\frac{b}{1-a})} \ .$$

Here the fixed or equilibrium point $\eta^\star$ is given by $C^{b/(1+b-a)}$. Substituting this for $\eta^\star$ we obtain

$$\check{F}'(\eta)\Big|_{\eta=\eta^\star} = \frac{-b}{1-a} \ .$$

Therefore, if $a+b > 1$, then $\check{F}'(\eta^\star) < -1$ and the system is unstable for sufficiently large T from the linear stability analysis in subsection 3.4. A numerical example illustrating this is provided in subsection 6.2.

5.2 General network with multiple heterogeneous users case

Suppose that the utility functions of the users are of the form given by (29) and the resource price functions are of the type described by (31). The utility function of user i is parametrized by $a_i \in (-\infty, 1)$, and the price function of resource l has parameter $b_l > 0$. Making use of the fact stated in subsection 4.3 that the stability of the map $\hat{F}$ defined in (28) is sufficient for convergence with both a homogenous delay and heterogenous delays, we only consider the case described in subsection 4.3 with a homogeneous delay T in the reverse path only. In this case the map $\hat{F}$ is given by

$$\hat{F}_l(\overline{\eta}) \ = \ \sum_{i \in I_l} \left(\sum_{j \in r_i} \left(\frac{\eta_j}{C_j} \right)^{b_j} \right)^{-\frac{1}{1-a_i}} \ , \quad l \in \mathcal{L} \ . \tag{35}$$

Note that $\hat{F}_l(\overline{\eta})$ is strictly decreasing in each of $\eta_j, j \in \cup_{i \in I_l} r_i$.

We define $b^i_{max} = \max_{l \in r_i} b_l$ for all $i \in \mathcal{I}$. Fix some finite positive constant $\overline{\alpha}$ larger than one. Suppose that $E_0 = \times_{l \in \mathcal{L}} E^l_0$, where $E^l_0 = [\overline{\beta}\eta^\star_l\ ,\ \overline{\alpha}\eta^\star_l]$, with $\overline{\beta}$ being a positive constant that satisfies the following component-wise inequalities

$$\hat{F}(\overline{\beta} \cdot \overline{\eta}^\star) < \overline{\alpha} \cdot \overline{\eta}^\star \ \text{ and } \ \overline{\beta} \cdot \overline{\eta}^\star < \hat{F}(\overline{\alpha} \cdot \overline{\eta}^\star) \ . \tag{36}$$

Lemma 2 *Suppose that $a_i + b^i_{max} < 1$ for all $i \in \mathcal{I}$. Define $\sigma = -\max_{i\in\mathcal{I}}\left\{\frac{b^i_{max}}{1-a_i}\right\} - \varepsilon$, where $0 < \varepsilon < 1 - \max_{i\in\mathcal{I}} \frac{b^i_{max}}{1-a_i}$. Then, any $\overline{\beta}$ such that $\overline{\alpha}^{1/\sigma} < \overline{\beta} < \overline{\alpha}^{\sigma}$ satisfies (36).*

Proof: The proof is given in Appendix D. ∎

We assume that $\overline{\beta}$ satisfies the condition in Lemma 2. Now, for $k = 1, 2, \cdots$, we define

$$E_k \quad = \quad \begin{cases} \prod_{l\in\mathcal{L}}[\overline{\alpha}^{\sigma^k}\eta^\star_l \ , \ \overline{\beta}^{\sigma^k}\eta^\star_l] \ , & k \text{ odd} \\ \prod_{l\in\mathcal{L}}[\overline{\beta}^{\sigma^k}\eta^\star_i \ , \ \overline{\alpha}^{\sigma^k}\eta^\star_l] \ , & k \text{ even} \end{cases} .$$

Lemma 3 *Suppose that $a_i + b^i_{max} < 1$ for all $i \in \mathcal{I}$. Then, $\hat{F}(E_{k-1}) \subset int(E_k) \subset E_k \subset int(E_{k-1})$, and $\cap_{k=0}^{\infty} E_k = \{\overline{\eta}^\star\}$.*

Proof: The proof is provided in Appendix E. ∎

Theorem 6 *Suppose that $a_i + b^i_{max} < 1$ for all $i \in \mathcal{I}$. If the initial function ϕ lies in $C([-T, 0], E_0)$, then $\overline{\eta}(t; \phi)$ produced by (27) converges to $\overline{\eta}^\star$ as $t \to \infty$ for all $T > 0$ and $\kappa_l > 0, l \in \mathcal{L}$.*

Proof: The theorem follows from Lemma 3 and Theorem 4. ∎

Now note that as $\overline{\alpha} \uparrow \infty$, $\hat{F}(\overline{\alpha}\cdot\overline{\eta}^\star) \to \underline{0} = [0, \ldots, 0]^T$. Hence, we can see that starting from any positive continuous initial function, the resource prices converge to $\overline{\mu}^\star$ as $t \to \infty$ from the above results because we can select sufficiently large $E_0 \subset \mathbb{R}^L_+$ that contains the initial function.

5.3 Comparison with the primal algorithm

In this subsection we comment on the difference in the conditions for convergence under the primal algorithm [28, 29] and the dual algorithm studied in this paper. The convergence condition of the primal algorithm with a single user and a single resource is first studied in [28], and a necessary and sufficient condition for convergence with an arbitrary delay using the utility and resource price functions of (29) and (31) is provided. The derived convergence condition states that user rate converges if and only if $a + b < -1$.[10] Since $b > 0$, this implies that the parameter of the utility function needs to be strictly smaller than -1. Clearly, this is a more restrictive condition than the one provided for the dual algorithm in this paper (i.e., ($a + b < 1$), which allows positive values of a for $b < 1$).

Similarly, in a general network case the sufficient conditions for convergence under the primal algorithm are given by $a_i + b^i_{max} < -1$ for all $i \in \mathcal{I}$ [29, 32], whereas the conditions that $a_i + b^i_{max} < 1$ for all $i \in \mathcal{I}$ suffice in the dual algorithm. Hence, the derived sufficient conditions for the primal algorithm in [29, 32] are more restrictive than those for the dual algorithm.

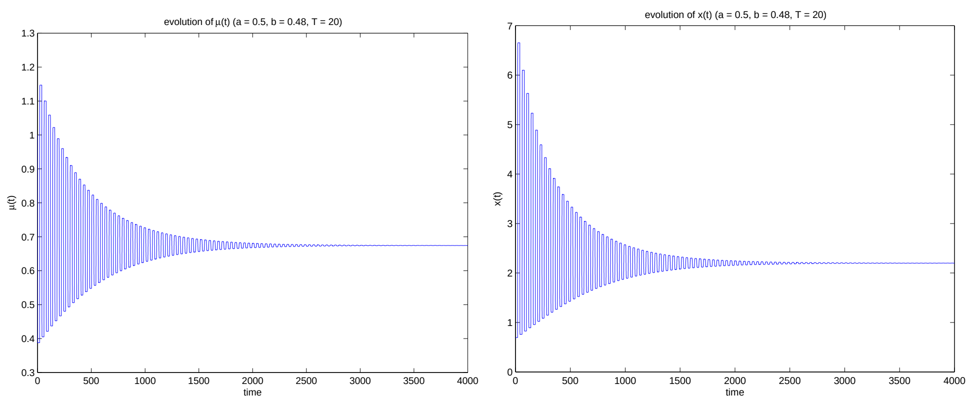
[10] In [28] we only considered the utility functions with $a < 0$ because when $a \geq 0$ the system is unstable for sufficiently large delays.

6 Numerical Result

In this section we present a numerical example to validate our results in the previous sections. We consider a single user, single resource case with the utility and resource price functions given in Section 5 with $a = 0.5$. The capacity of the resource is set to $C = 5$. We vary the resource price function parameter b to create both a stable scenario and an unstable scenario according to our condition in Theorem 5.

6.1 Stable system

In the first case we set $b = 0.49$. Since $a + b = 0.99 < 1$, Theorem 5 tells us that the resource price and user rate converge irrespective of the gain κ and the delay T. For the numerical example the delay is set to $T = 200$ and the gain is set to $\kappa = 1$. The initial value $\mu(t)$ is set to 1.2 for all $t \in [-T, 0]$. The evolution of $\mu(t)$ and $x(t)$ is plotted in Fig. 4. As one can see both $\mu(t)$ and $x(t)$ converge to their equilibrium values of 0.6715 and 2.218, respectively.



(a) (b)

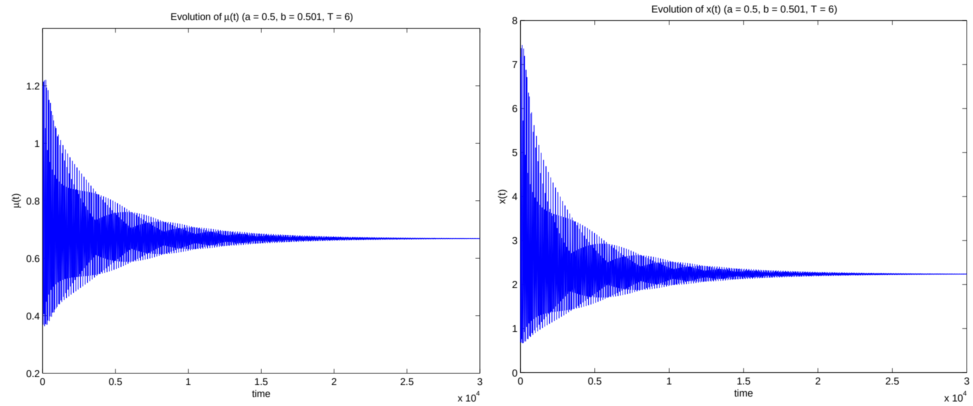
Figure 4: evolution of $\mu(t)$ and $x(t)$ ($a =$ 0.5, $b =$ 0.49, $T =$ 200). (a) $\mu(t)$, (b) $x(t)$.

6.2 Unstable system

In the second example we have increased the resource price function parameter b to 0.501. Since $a + b = 1.001 > 1$, the system loses its stability for sufficiently large delay T. The linear stability analysis provided in subsection 3.4 tells us that the linearized system in (21) is stable if and only if

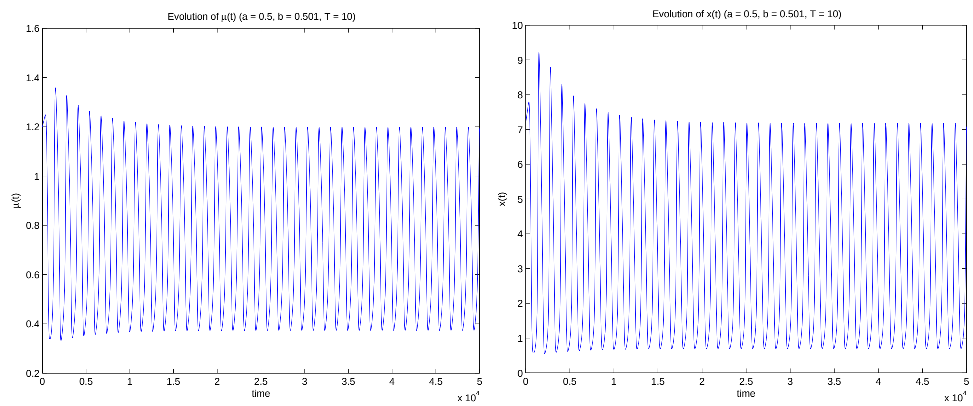
$$T \quad \leq \quad \frac{\cos^{-1}\left((\check{F}'(\eta^\star))^{-1}\right)}{\zeta(\eta^\star)\sqrt{(\check{F}'(\eta^\star))^2 - 1}} = 7.28 \ .$$

Figs. 5 and 6 plot the evolution of $\mu(t)$ and $x(t)$ for $T = 6$ and $T = 10$, respectively, sampled at every 20 unit times. As one can easily see, the system with $T = 6$ is stable and $\mu(t)$ and $x(t)$ converge to the equilibrium points of 0.6685 and 2.2379, respectively. However, when the delay T is increased to 10, the system loses stability and exhibits oscillatory behavior as shown in Fig. 6.



(a) (b)

Figure 5: evolution of $\mu(t)$ and $x(t)$ ($a = 0.5$, $b = 0.501$, $T = 6$). (a) $\mu(t)$, (b) $x(t)$.



(a) (b)

Figure 6: evolution of $\mu(t)$ and $x(t)$ ($a = 0.5$, $b = 0.501$, $T = 10$). (a) $\mu(t)$, (b) $x(t)$.

7 Conclusions

We studied the issue of convergence of user rates and resource prices under a dual algorithm in the presence of communication delays. Using the same framework first employed in [29] for the primal algorithm, we derived sufficient conditions for convergence with arbitrary delays. In addition, we showed that these sufficient conditions can be obtained from a simple underlying discrete time system. We applied our result to an example of popular utility and resource price functions and derived sufficient conditions for convergence. In the simpler case of a single user utilizing a resource, we derived the necessary and sufficient condition for convergence. In addition, we studied the case when the convergence condition is violated and, using a linear stability analysis, provided an upper bound on the delay for convergence. Numerical examples are presented to validate our analysis. We believe that the framework used in this paper as well as in [29] is quite general and will prove to be useful for studying the convergence property of a variety of distributed control systems, in particular in the context of networking and networked control systems.

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A Proof of Theorem 1

The proof of the theorem is based on the following Lemma 4.

Lemma 4 *[7, p. 507] Suppose that $I^\star = [a^\star, b^\star] \subset \mathbb{R}$, where $a^\star < b^\star$, is a compact interval and $\xi : \overline{\mathbb{R}}_+ \to I^\star$ is a continuous function. If $\sigma : \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ is a bounded, continuous, strictly positive function and $u(t)$ is a solution of following equation*

$$\sigma(t)\dot{u}(t) + u(t) = \xi(t) \ , \tag{37}$$

with $u(0) \in I^\star$, then $u(t) \in I^\star$ for all $t \geq 0$.

Proof: The existence of a unique solution of (37) is guaranteed by the theorem in [27, p. 74]. We will prove this lemma by contradiction. Suppose that the lemma is not true. Define

$$t_0 = \inf\{t \geq 0 \mid u(t) \notin I^\star\} \ .$$

First, suppose that $u(t_0) = b^\star$. Then, every interval $(t_0, t_0 + \delta), \delta > 0$, contains a point τ such that $u(\tau) > b^\star$ and $\dot{u}(\tau) > 0$. However, if $u(\tau) > b^\star$, (37) tells us that $\dot{u}(\tau) < 0$ because $\xi(t) \leq b^\star$, which is a contradiction. The case $u(t_0) = a^\star$ can be shown to lead to a similar contradiction. This completes the proof. ∎

We proceed with the proof of Theorem 1. Apply Lemma 4 to $\frac{d}{dt}\omega_0(t) = -\alpha\omega_0(t) + \alpha\check{F}(\eta(t-T))$. Clearly, if $\omega_0(0) \in J$ and initial function $\phi \in Y_J$, then $\omega_0(t) \in J$ for all $0 \leq t \leq T$. By applying Lemma 4 to $\frac{d}{dt}\omega_1(t) = -\alpha\omega_1(t) + \alpha\omega_0(t)$, we can argue that $\omega_1(t) \in J$ for all $0 \leq t \leq T$. Following this recursive argument, we can show that $\omega_i(t) \in J$ for $i = 0, \ldots, r$ and $\eta(t) \in J$ for all $0 \leq t \leq T$. Now by an induction argument on time (called the method of steps [5]) the same can be argued for all $t \geq 0$.

B Proof of Theorem 2

The proof of the theorem is a simple application of the following lemma.

Lemma 5 *Consider the same setup in Lemma 4. Assume that $\bar{I} = [\bar{a}, \ \bar{b}]$ is a compact interval whose interior contains $I^\star$, i.e., $I^\star \subset int(\bar{I})$. Then, for any $u(0) = u_0 \in \mathbb{R}_+$, there exists finite $t_0 := t_0(u_0, \bar{I})$ such that $u(t) \in \bar{I}$ for all $t \geq t_0$.*

Proof: First, note that if $u(t^*) \in \bar{I}$ for some $t^* \geq 0$, then from Lemma 4 $u(t) \in \bar{I}$ for all $t \geq t^*$. Thus, we only need to show that there exists some finite t_0 such that $u(t_0) \in \bar{I}$. Suppose that this is not true. First, assume that $u(t) > \bar{b}$ for all $t \geq 0$. Then, from (37) we have $\dot{u}(t) = \frac{\xi(t)-u(t)}{\sigma(t)} \leq \frac{b^\star - \bar{b}}{\sigma(t)}$. Since $\sigma(t)$ is bounded, we can find a positive ε such that $\dot{u}(t) \leq -\varepsilon$ for all $t \geq 0$. However, this implies that $u(t) \downarrow -\infty$, which contradicts the assumption that $u(t) > \bar{b}$ for all $t \geq 0$. The other case $u(t) < \bar{a}$ for all $t \geq 0$ can be shown to lead to a similar contradiction, and the lemma follows. ∎

We now proceed with the proof of the theorem. First, since $\check{F}(J_0) \subset int(J_1)$, we can find a set of compact intervals $\{L_1^i, i = 0, \ldots, r\}$ such that

$$\check{F}(J_0) \subset int(L_1^0) \subset L_1^0 \subset int(L_1^1) \subset \cdots \subset int(L_1^r) \subset L_1^r \subset int(J_1)\ . \tag{38}$$

Using the property in (38), we can repeatedly apply Lemma 5, starting with the third equation in (19) for $\omega_0(t)$ and then the second equation with $\omega_i(t)$, $i = 1, \ldots, r$, to find finite $t_1^i, i = 0, \ldots, r$, where $0 \leq t_1^0 \leq t_1^1 \leq \ldots \leq t_1^r$, such that $\omega_i(t) \in L_1^i$ for all $t \geq t_1^i$. Finally, applying Lemma 5 to the first equation in (19) we can find finite $t_1^\star \geq t_1^r$ such that $\eta(t) \in J_1$ for all $t \geq t_1^\star$.

Now, by an induction argument for each $k = 2, 3, \ldots$, one can find an increasing sequence $t_k^\star, k = 1, 2, \ldots$, such that, for all $t \geq t_k^\star$, $\eta(t) \in J_k$ and $\omega_i(t) \in J_k, i = 0, 1, \ldots, r$. Now the theorem follows from the assumption that diam(J_k) $\to 0$ as $k \to \infty$ and $\cap_{k \geq 1} J_k = \{q_N(\mu^\star)\}$.

C Proof of Lemma 1

In this proof we only consider the case of even k. Proof for the case with odd k follows in a similar manner. From the monotonicity of the map $\overline{F}(\mu) = C^{-b}\mu^{-b/(1-a)}$, it suffices to show that $\overline{F}(\overline{\beta}^{\sigma^k}\mu^\star) \in I_{k+1}$ and $\overline{F}(\overline{\alpha}^{\sigma^k}\mu^\star) \in I_{k+1}$.

First,

$$\begin{aligned}
\overline{F}(\overline{\alpha}^{\sigma^k}\mu^\star) &= C^{-b}(\overline{\alpha}^{\sigma^k}\mu^\star)^{-b/(1-a)} = C^{-b}\mu^{\star -b/(1-a)}\overline{\alpha}^{\sigma^k(-b/(1-a))} \\
&= \mu^\star\overline{\alpha}^{\sigma^k(-b/(1-a))} > \mu^\star\overline{\alpha}^{-\sigma^{k+1}}\ ,
\end{aligned}$$

where the last equality follows from the fact that $\mu^\star$ is a fixed point of the map $\overline{F}$, and the inequality follows from the assumption that $0 < \frac{b}{1-a} < \sigma < 1$ and $\overline{\alpha} > 1$. Clearly, $\mu^\star\overline{\alpha}^{\sigma^k(-b/(1-a))} < \mu^\star < \mu^\star\overline{\beta}^{-\sigma^{k+1}}$ because $\overline{\beta} < 1 < \overline{\alpha}$.

Similarly,

$$\begin{aligned}
\overline{F}(\overline{\beta}^{\sigma^k}\mu^\star) &= C^{-b}(\overline{\beta}^{\sigma^k}\mu^\star)^{-b/(1-a)} = C^{-b}\mu^{\star -b/(1-a)}\overline{\beta}^{\sigma^k(-b/(1-a))} \\
&= \mu^\star\overline{\beta}^{\sigma^k(-b/(1-a))} < \mu^\star\overline{\beta}^{-\sigma^{k+1}}\ .
\end{aligned}$$

Therefore, $\overline{F}(I_k) \subset int(I_{k+1})$ and the lemma follows.

D Proof of Lemma 2

From (35) we have

$$\check{F}_l(\overline{\alpha}\cdot\overline{\eta}^\star) = \sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\frac{\overline{\alpha}\cdot\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}}$$

Thus,

$$\begin{aligned}
\check{F}_l(\overline{\alpha}\cdot\overline{\eta}^\star) &= \sum_{i\in I_l}\left(\sum_{j\in r_i}\overline{\alpha}^{b_j}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&\geq b\sum_{i\in I_l}\left(\sum_{j\in r_i}\overline{\alpha}^{b^i_{max}}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \sum_{i\in I_l}\overline{\alpha}^{-\frac{b^i_{max}}{1-a_i}}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&> \sum_{i\in I_l}\overline{\alpha}^{\sigma}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \overline{\alpha}^{\sigma}\sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \overline{\alpha}^{\sigma}\eta_l^\star\,.
\end{aligned}$$

where the second inequality follows from the assumption $\sigma < -\frac{b^i_{max}}{1-a_i}$ for all $i\in\mathcal{I}$. Therefore, if $\overline{\beta} < \overline{\alpha}^\sigma$, then $\overline{\beta}$ satisfies the second condition in (36).

Similarly,

$$\begin{aligned}
\check{F}_l(\overline{\beta}\overline{\eta}^\star) &= \sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\frac{\overline{\beta}\cdot\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&\leq \sum_{i\in I_l}\left(\sum_{j\in r_i}\overline{\beta}^{b^i_{max}}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \sum_{i\in I_l}\overline{\beta}^{-\frac{b^i_{max}}{1-a_i}}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&< \sum_{i\in I_l}\overline{\beta}^{\sigma}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \overline{\beta}^{\sigma}\sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}}
\end{aligned}$$

$$
\begin{aligned}
&< \quad \overline{\alpha}\sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \quad \overline{\alpha}\cdot\eta_l^\star ,
\end{aligned}
$$

where the first inequality follows from the assumption $\overline{\beta} < 1$. Hence, if $\overline{\beta}^{\sigma} < \overline{\alpha}$ (or $\overline{\beta} > \overline{\alpha}^{1/\sigma}$), then $\check{F}(\overline{\beta}\cdot\overline{\eta}^\star) < \overline{\alpha}\cdot\overline{\eta}^\star$, and the first condition in (36) holds.

E Proof of Lemma 3

We first show that $E_{k+1} \subset int(E_k)$, for all $k = 0, 1, \cdots$. This can be easily shown as follows. From the assumption we have $\overline{\beta} < \overline{\alpha}^{\sigma}$ and $\overline{\beta}^{\sigma} < \overline{\alpha}$. Hence, $E_1 \subset int(E_0)$. Now similarly as before, since $\overline{\beta} < \overline{\alpha}^{\sigma}$ we have $\overline{\alpha}^{\sigma^2} < \overline{\beta}^{\sigma}$, and because $\overline{\beta}^{\sigma} < \overline{\alpha}$, $\overline{\alpha}^{\sigma} < \overline{\beta}^{\sigma^2}$, which follows from the fact that $-1 < \sigma < 0$. By repeating this we get (i) $\overline{\alpha}^{\sigma^{k+1}} < \overline{\beta}^{\sigma^k}$ and $\overline{\alpha}^{\sigma^k} < \overline{\beta}^{\sigma^{k+1}}$ for odd k and (ii) $\overline{\beta}^{\sigma^{k+1}} < \overline{\alpha}^{\sigma^k}$ and $\overline{\beta}^{\sigma^k} < \overline{\alpha}^{\sigma^{k+1}}$ for even k, proving that $E_{k+1} \subset int(E_k)$. The claim that $\cap_{k=0}^{\infty} E_k = \{\overline{\eta}^\star\}$ follows trivially from the fact that $\lim_{k\to\infty}\sigma^k = 0$ because $|\sigma| < 1$. Hence, $\lim_{k\to\infty}\overline{\alpha}^{\sigma^k} = 1 = \lim_{k\to\infty}\overline{\beta}^{\sigma^k}$.

Now we prove that $\check{F}(E_k) \subset int(E_{k+1}), k = 0, 1, \cdots$. Here we prove this only for the case when k is even. The other case can be proved similarly. Following the same approach in the proof of Lemma 2 we have

$$
\check{F}_l(\overline{\alpha}^{\sigma^k}\overline{\eta}^\star) \quad = \quad \sum_{i\in I_l}\left(\sum_{j\imath\overline{\alpha}^{\sigma^k}nr_i}\Big(\frac{\overline{\alpha}^{\sigma^k}\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \quad .
$$

Since $\overline{\alpha}^{\sigma^k} > 1$ (because k is even),

$$
\begin{aligned}
\check{F}_l(\overline{\alpha}^{\sigma^k}\overline{\eta}^\star) \quad &= \quad \sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\overline{\alpha}^{\sigma^k}\Big)^{b_j}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&\geq \quad b\sum_{i\in I_l}\left(\sum_{j\in r_i}\Big(\overline{\alpha}^{\sigma^k}\Big)^{b^i_{max}}\Big(\frac{\eta_j^\star}{C_j}\Big)^{b_j}\right)^{-\frac{1}{1-a_i}} \\
&= \quad \sum_{i\in I_l}\Big(\overline{\alpha}^{\sigma^k}\Big)^{-\frac{b^i_{max}}{1-a_i}}\left(\sum_{j\in r_i}\left(\frac{\eta_j^\star}{C_j}\right)\right)^{-\frac{b_j}{1-a_i}} \\
&> \quad \sum_{i\in I_l}\Big(\overline{\alpha}^{\sigma^k}\Big)^{\sigma}\left(\sum_{j\in r_i}\left(\frac{\eta_j^\star}{C_j}\right)\right)^{-\frac{b_j}{1-a_i}} \\
&= \quad \overline{\alpha}^{\sigma^{k+1}}\sum_{i\in I_l}\left(\sum_{j\in r_i}\left(\frac{\eta_j^\star}{C_j}\right)\right)^{-\frac{b_j}{1-a_i}} \\
&= \quad \overline{\alpha}^{\sigma^{k+1}}\eta_l^\star .
\end{aligned}
$$

where the strict inequality follows from the assumption that $\sigma < -\frac{b^i_{max}}{1-a_i}$ for all $i \in \mathcal{I}$, and the last equality follows from the assumption that $\overline{\eta}^\star$ is the fixed point of the map $\check{F}(\cdot)$. Similarly one can show that $\check{F}_l(\overline{\beta}^{\sigma^k}\overline{\eta}^\star) < \overline{\beta}^{\sigma^{k+1}}\eta^\star_l$. This completes the proof of the lemma.

F Proof of Theorem 3

We first introduce a lemma that will be used in the proof of the theorem.

Lemma 6 *(Invariance) Suppose that $E = \times_{l\in\mathcal{L}}E_l$ is a closed, convex set that is invariant under the map F. Then, if the initial function $\phi \in C([-T_{max}, 0], E) =: Y_E$, then $\overline{\eta}(t;\phi) \in E$ for all $t \geq 0$.*

Proof: We prove the lemma by contradiction. Suppose that the claim is false. Then, there exist some initial function $\phi \in Y_E$ and $t,\ t \geq 0$, such that $\overline{\eta}(t) \notin E$. Define

$$t_0 \;=\; \inf\{t \geq 0 \mid \text{ every interval } [t,\ t'),\ t' > t,\ \exists\, t_1,\ t < t_1 < t',\ \text{such that } \overline{\eta}(t_1) \notin E\}\ .$$

Then, there is $l \in \mathcal{L}$ such that for all (t_0, t'), where $t' > t_0$, there exists $\tilde{t}_0, t_0 < \tilde{t}_0 < t'$, such that $\eta_l(\tilde{t}_0) \notin E_l$. We assume that $\eta_l(t)$ leaves through the right end, i.e., $\eta_l(t_0) = \sup\ E_l$. Then, for all (t_0, t') there exists $\tilde{t}, t_0 < \tilde{t} < t'$, such that $\eta_l(\tilde{t}) > \sup\ E_l$ and $\frac{d}{dt}\eta_l(\tilde{t}) > 0$. This, however, leads to a contradiction as follows. From (24) we have

$$\frac{d}{dt}\eta_l(t) \;=\; \kappa_l \cdot g_l'(g_l^{-1}(\eta_l(t)))\,(F_l(\tilde{\eta}(t)) - \eta_l(t)) < 0$$

because $\kappa_l \cdot g_l'(g_l^{-1}(\eta_l(t))) > 0$ and $F_l(\tilde{\eta}(t)) \in E_l$ and, hence, is less than or equal to $\sup E_l$ $(< \eta_l(\tilde{t}))$, which contradicts the earlier assumption that $\frac{d}{dt}\eta_l(\tilde{t}) > 0$. The other case that $\eta_l(t)$ leaves E_l through the left end, i.e., $\eta_l(t_0) = \inf\ E_l$, can be shown to lead to a similar contradiction. Therefore, the lemma follows. ■

In the rest of the proof of Theorem 3 we omit the dependency on ϕ since there is no confusion. Our proof utilizes the following lemmas.

Lemma 7 *Fix $k, k \geq 0$. Let $\tilde{E} = \times_{l\in\mathcal{L}}\tilde{E}_l$ be an open, convex set that contains $F(E^\Xi_k)$ and whose closure is contained in $int(E_k)$, i.e., $cl(\tilde{E}) \subset int(E_k)$. Suppose that the initial function $\phi \in C([-T_{max}, 0], E_k)$. Then, there exists a finite $\tilde{t},\ \tilde{t} \geq 0$, such that, for all $t \geq \tilde{t}$, $\overline{\eta}(t) \in \tilde{E}$.*

In order to prove the lemma, we first prove the following coordinate-wise invariance

Lemma 8 *(Coordinate-wise Invariance) If $\eta_l(\tilde{t}) \in \tilde{E}_l$ for some $\tilde{t} \geq 0$, then $\eta_l(t) \in \tilde{E}_l$ for all $t \geq \tilde{t}$.*

Proof: Suppose that the lemma is not true, and there exists $\bar{t} > \tilde{t}$ at which $\eta_l(\bar{t}) = \inf \tilde{E}_l$ or $\eta_l(\bar{t}) = \sup \tilde{E}_l$. We assume that $\bar{t}$ is the smallest such time and $\eta_l(\bar{t}) = \sup \tilde{E}_l > \ \sup\ F_l(E^\Xi_k)$. Then, we can find $\bar{t}_1 < \bar{t}$ such that for all $t \in (\bar{t}_1, \bar{t})$, $\eta_l(t) \in \tilde{E}_l \setminus F_l(E^\Xi_k)$. This implies that $\frac{d}{dt}\eta_l(t) < 0$ for all $t \in (\bar{t}_1, \bar{t})$ from (24) because $F_l(\tilde{\eta}(t)) \leq \sup\ F_l(E_k^\Xi)$ and, thus, $\eta_l(\bar{t}) < \sup\ \tilde{E}_l$, leading to a contradiction. A similar argument can be used for the case $\eta_l(\bar{t}) = \inf\ \tilde{E}_l$. ■

Now let us proceed with the proof of Lemma 7.

Proof: Suppose that the lemma is false. Then, from Lemma 8 there exists $l \in \mathcal{L}$ such that for all $t \geq 0$, $\eta_l(t) \notin \tilde{E}_l$. We show that this leads to a contradiction. Suppose that $\eta_l(t) \geq \sup\ \tilde{E}_l$ for all $t \geq 0$. Then, one can see that $\frac{d}{dt}\eta_l(t) < 0$ from (24). Combined with the assumption that $\eta_l(t) \geq \sup\ \tilde{E}_l$ for all $t \geq 0$, this implies that $\eta_l(t)$ converges to some $\bar{\eta}_l \geq \sup\ \tilde{E}_l$. Since $\sup\ \tilde{E}_l > \sup\ F_l(E_k^\Xi)$ with $\delta := \sup\ \tilde{E}_l - \sup\ F_l(E_k^\Xi) > 0$, there exists some positive constant ε such that $\frac{d}{dt}\eta_l(t) \leq -\varepsilon \cdot \delta < 0$ for all sufficiently large t from (24). This, however, implies that $\eta_l(t) \downarrow -\infty$ as $t \uparrow \infty$, contradicting the assumption that $\eta_l(t) \geq \sup\ \tilde{E}_l$ for all $t \geq 0$. A similar contradiction can be shown when we assume $\eta_l(t) \leq \inf\ \tilde{E}_l$ for all $t \geq 0$. This completes the proof of the lemma. ■

Lemma 9 *Let $E = \times_{l \in \mathcal{L}} E_l$ be a closed set invariant under F and $\tilde{E} = \times_{l \in \mathcal{L}} \tilde{E}_l$ an open, convex set that contains $F(E^\Xi)$ and whose closure is contained in $int(E)$, i.e., $cl(\tilde{E}) \subset int(E)$. Suppose that the initial function $\phi \in C([-T_{max}, 0], E)$ and $\overline{\eta}(t_1) \in \tilde{E}$ for some $t_1 \geq 0$. Then, $\overline{\eta}(t) \in \tilde{E}$ for all $t \in [t_1,\ t_1 + T_{max}]$.*

Proof: The lemma follows directly from Lemma 8. ■

Now we are ready to proceed with the proof of Theorem 3. By repeatedly applying Lemmas 7, 9 and 6 one can find a sequence of finite $t_k, k = 1, 2, \ldots$, such that $\overline{\eta}(t) \in E_k$ for all $t \geq t_k$. The theorem now follows from the assumption that $\cap_{k=1}^{\infty} E_k = \{\overline{\eta}^\star\}$.

G Proof of Theorem 4

The proof of Theorem 4 is essentially identical to that of Theorem 3: One can easily verify that the same invariance properties in Lemmas 6 and 8 hold for (27) following the same arguments. Moreover, following the same proof of Lemma 7, we can show that for every $k \geq 0$ and any initial function $\phi \in C([-T, 0], \check{E}_k)$, there exists finite $\bar{t}_{k+1} \geq 0$ such that, for all $t \geq \bar{t}_{k+1}$, $\overline{\eta}(t) \in \check{E}_{k+1}$. Therefore, as in the proof of Theorem 3, we can find a sequence of $\bar{t}_k, k = 1, 2, \cdots$, such that $\overline{\eta}(t) \in \check{E}_k$ for all $t \geq \bar{t}_k$, and the convergence of $\overline{\eta}(t)$ under (27) follows from the assumption that $\cap_{k=1}^{\infty} \check{E}_k = \{\overline{\eta}^\star\}$.